modeling and numerical simulation

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Chapter 1

Sample Space and Random Variable

1.1 Sample Space and Events

1.1.1 Random Experiments

In the study of probability, any process of observation is referred to as an experiment. The results of an observation are called outcomes of the experiment. An experiment is called a random experiment if its outcomes cannot be predicted in advance with certainty. Some examples of a random experiment are the roll of a die, the toss of coin, drawing a card from a deck, or selecting a message signal for transmission from several messages ... .

1.1.2 Sample Space

The set of all possible outcomes of a random experiment is called **the sample space** of the experiment, and it is denoted by Ω.
An element of the sample space Ω is called a sample point and each outcome of a random experiment corresponds to a sample point.

Exemple 1: Find the sample space for the experiment of tossing a coin in the two following cases

1. Tossing a coin once.
2. Tossing a coin twice.

Solution:

- In the first case there are two possible outcomes, heads or tails. Thus

$$\Omega = \{H, T\}. \tag{1.1}$$

- In the second case, there are four possible outcomes, which are the pairs of heads and tails, viz

$$\Omega = \{HH, HT, TH, TT\}. \tag{1.2}$$

Where H and T represent head and tail respectively.

Exemple 2: Find the sample space for the experiment of tossing a coin repeatedly and of counting the number of tosses required untilthe first head appears.

Solution: It is clear that all possible outcomes for this experiment are the terms of the sequence $1, 2, 3, ...$ Thus

$$\Omega = \{1, 2, 3, ...\}. \tag{1.3}$$

Note that there are an infinite numbers of outcomes.

Exemple 3: Find the sample space for the experiment of measuring (in hours) the life time of transistor.

Solution: It is easy and to note that all possible outcomes are non-negative real numbers, viz

$$\Omega = \{\tau : 0 \leq \tau \leq \infty\}. \tag{1.4}$$

Remarks:

1. Any particular experiment can often have many different sample spaces depending on the observation of interest (examples 1 and 2).
2. A sample space Ω is said to be discrete if consists of a finite number of sample points (as in example 1) or infinite countably sample points (as in the example 2).
3. A set is called countable if the its elements can be placed in one-to-one correspondence with the positive integers.
4. A sample space is called to be continuous if the sample points is constitute a continuum (as in example 3).

1.1.3 Events

Since we have identified a sample space Ω as the set of all possible outcomes of a random experiment, then any subset of the sample space Ω is called an **event**.

Example 4: Consider the experiment of example 2, and find the events A, B and C, Where

- A is the event that the number of tosses required until the first head appears is even.
- B is the event that the number of tosses required until the first appears is odd.
- Finally, the event C represents the number of tosses required until the first head appears is less then 7.

Solution: The events A,B and C are

$$A = \{2, 4, 6, 8, ...\}. \tag{1.5}$$

$$B = \{1, 3, 5, 7, ...\}. \tag{1.6}$$

$$B = \{1, 2, 3, 4, 5, 6\}. \tag{1.7}$$

Remarks:

- The empty set $\emptyset$ is an event for any sample space.
- The entire sample space is also an event for any sample space.
- A given event is said to have occurred if the outcomes of the of the experiment is one of the outcomes of the event.

Mutually Exclusive Events

If some events con never occur together is to be mutually exclusive. For example, It is impossible that a coin can come up both heads and tails.

Definition:

- The Two events A and B are mutually exclusive if they have no outcomes in common.
- More generally, a collection of events $A_1, A_2, A_3, ..., A_N$ is said to be mutually exclusive if no to of them have any outcomes in common.

Example 5: Let the events to be the events A and B of the example 4. Is it possible for events A and B both to occur?.

Solution: It is impossible for A and B both to occur, because these events are mutually exclusive.

1.2 Random Variables

In many probabilistic experiments the outcomes are numerical. In other probabilistic experiments the outcomes are not numerical, but they may be associated with values of interest. For example, if the experiment is the selection of student fro a given class, we may associate to each outcome (student) his weight or his height which is a numerical value. When dealing with such numerical values in such random experiment, it is often useful to assign probabilities to them. This procedure is done trough the notation of random variable.

1.2.1 Definitions

For any random experiment we define a set of outcomes which is the sample space Ω.

Random Variable Definition: A random variable $X(\chi)$ is a function that assigns a single real number to each sample point χ (outcome) in a sample space Ω.

Remarks:

- Mostly, a letter X is used to express the random variable instead of the function $X(\chi)$.
- It is clear that a random variable is not a variable in the usual sense, but it is a function.
- The set of all possible outcomes of a random experiment forms what is called **the domain** of the random variable (sample space Ω).

- The collection of all numerical values (real numbers) of a random variable X forms what is called **the range** of the random variable, which is a certain subset of the set of all real numbers.

- It is possible to assign a same value of the random variable to more than one outcome (sample point), but the opposite is not true.

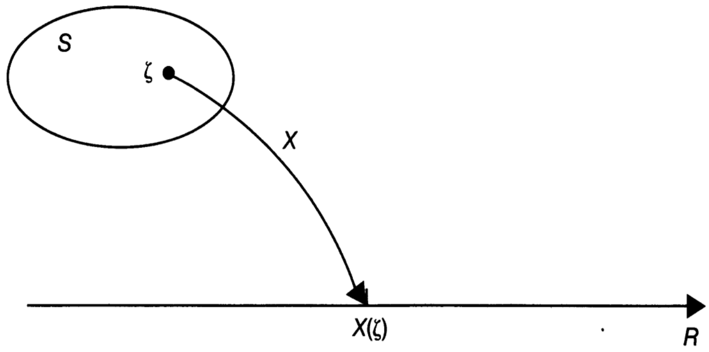

Fig. 2-1 Random variable *X* as a function.

Example 6: Consider a class of students, in which we are interested in measuring the weight of a student that randomly chosen from this class.

- How can you choose the random variable that represents this procedure?.
- What are the possible values of the random variable?.
- Is this random variable discrete or continuous?.

Solution:

- We choose the random variable W which assigns to each chosen student the value of his weight.
- The possible values of the random variable are all real numbers.
- This random variable is discrete.

1.2.2 Events Defined by Randdom Variable

If X is a random variable and x is a fixed real number, we can define the event $(X = x)$ as

$$(X = x) = \{\chi : X(\chi) = x\}. \tag{1.8}$$

By the same way, for fixed numbers x, x_1 and x_2, we can define the following events

$$(X \leq x) = \{\chi : X(\chi) \leq x\}. \tag{1.9}$$

$$(X > x) = \{\chi : X(\chi > x\}. \tag{1.10}$$

$$(x_1 < X \leq x_2) = \{\chi : (x_1 < X(\chi) \leq x\}. \tag{1.11}$$

The probabilities of these events are denoted by

$$P(X = x) = P\{\chi : X(\chi) = x\}. \tag{1.12}$$

$$P(X \leq x) = P\{\chi : X(\chi) \leq x\}. \tag{1.13}$$

$$P(X > x) = P\{\chi : X(\chi > x\}. \tag{1.14}$$

$$P(x_1 < X \leq x_2) = P\{\chi : (x_1 < X(\chi) \leq x\}. \tag{1.15}$$

Example 7: In the experiment of tossing a fair coin three times, the sample space Ω consists of eight equally likely sample points. If X is a random variable giving the number of heads obtained, find
a)$P(X = 2)$ *b*)$P(X \leq 2)$ *c*)$P(X > 2)$ *d*)$P(0 < X \leq 2)$

Solutions: The sample space is given by

$$\Omega = \{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH\}. \tag{1.16}$$

a)

$$A = (X = 2) = \{\chi : X(\chi) = 2\} = \{HHT, HTH, THH\} \subset \Omega.$$

Since the sample points are likely equally to occur, then the probability of the event A to occur is

$$P(A) = P(X = 2) = p(HHT) + p(HTH) + p(THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}.$$

b)

$$B = (X < 2) = \{\chi : X(\chi) < 2\} = \{HTT, TTT, TTH, THT\} \subset \Omega.$$

The probability of the event B to occur is

$$P(B) = P(X < 2) = p(HTT)+p(TTT)+p(TTH)+p(THT) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{4}{8} = \frac{1}{2}.$$

c)

$$C = (X \geq 2) = \{\chi : X(\chi) \geq 2\} = \{HHH, HHT, HTH, THH\} \subset \Omega.$$

The probability of the event C to occur is

$$P(C) = P(X \geq 2) = p(HHH)+p(HHT)+p(HTH)+p(THH) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{4}{8} = \frac{1}{2}.$$

d)

$$D = (0 < X \leq 2) = \{\chi : 0 < X(\chi) \leq 2\} = \{HHT, HTH, HTT, TTH, THT, THH\} \subset \Omega.$$

The probability of the event C to occur is

$$P(D) = P(X \geq 2) = p(HHT)+p(HTH)+p(HTT)+p(TTH)+p(THT)+p(THH) = \frac{6}{8} = \frac{3}{4}.$$

1.2.3 Discrete Random Variable

A random variable is said a discrete random variable if its possible values form a discrete set.

1.2.3.1 Probability Mass Function

The probability mass function of a discrete random variable X is the function given by

$$p(x) = P(X = x). \tag{1.17}$$

The probability mass function sometimes called the probability distribution.

Example 8: The number of flaws in a 1*centimeter* of copper wire manufactured by a certain process varies from wire to wire. Overall, 48% of the wires produced have no flaw, 39% have one flaws, 12% have two flaws, 1% have three flaws.
Let X be the random variable that represents the number of flaws in a randomly selected piece of wire.

1. Find the possible values of the random variable X.
2. Find the PMF of $X = x$ for $x = 0, 1, 2, 3$.
3. Plot the PMF of this random variable.

Solution :

1. The possible values of X are $x = 0, 1, 2, 3$.
2. PMF of $X = x$:

   $P(X = 0) = p(0) = 0.48, \quad P(X = 1) = p(1) = 0.39, \quad P(X = 2) = p(2) = 0.12,$
   $P(X = 3) = p(3) = 0.01, \quad P(X = x) = p(x) = 0$ For any value of x other than $x = 0, 1, 2, 3$.

3. The plot of the PMF

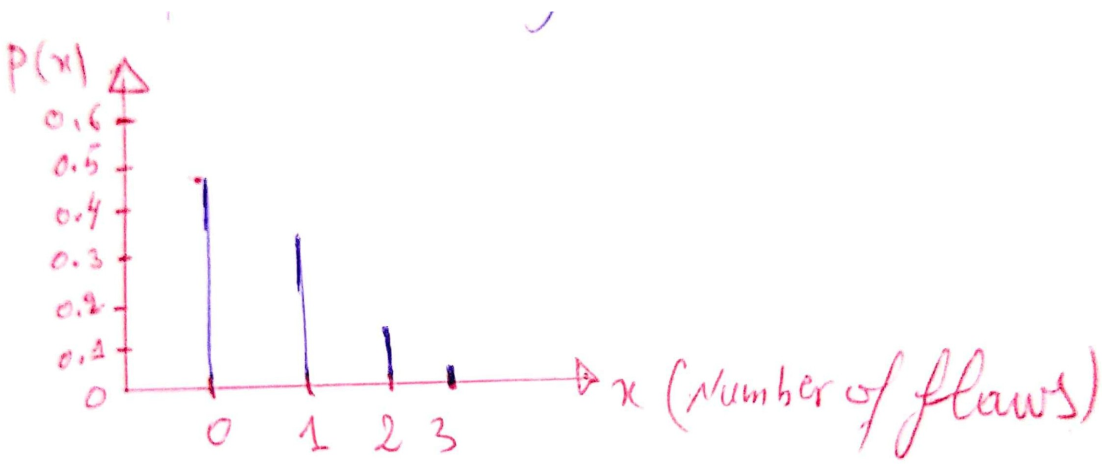

Figure 1.1: Probability mass function of X

Example 9: Repeat the example 7, and answer the questions of example 8.

Solustion :

1. The possible values of the random variable X are $x = 0, 1, 2, 3$.

2.

$$P(X=0) = p(0) = \frac{1}{8}, \quad P(X=1) = p(1) = \frac{3}{8}, \quad P(X=2) = p(2) = \frac{3}{8},$$
$$P(X=3) = p(3) = \frac{1}{8}, \quad P(X=x) = p(x) = 0 \text{ For any value of } x \text{ other than } x = 0, 1, 2, 3.$$

3. The plot of the PMF of the random variable X

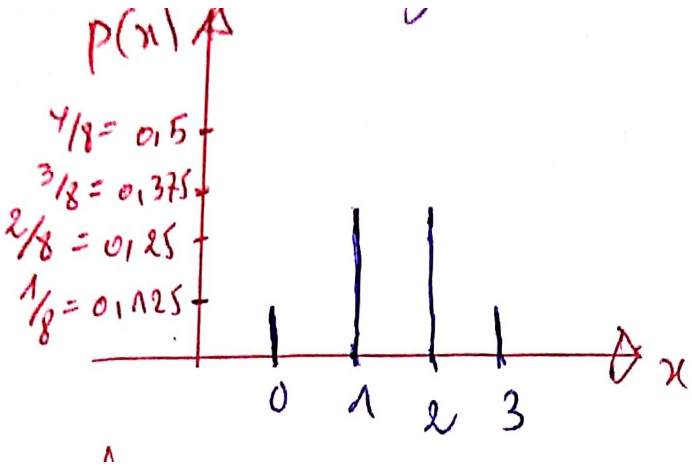

Figure 1.2: Probability mass function of X

Remark : The sum over the probabilities of all the possible values of the random variable X must be equal to one, viz

$$\sum_x P(X = x) = \sum_x p(s) = 1. \tag{1.18}$$

Example 10: Use the examples 8 and 9 and Check that the sum over the probabilities of all the possible values of the random variable is equal to 1.

Solution :

$$\sum_{x=0}^{3} p(x) = p(0) + p(1) + p(2) + p(3) = 0.48 + 0.39 + 0.12 + 0.01 = 1.$$

$$\sum_{x=0}^{3} p(x) = p(0) + p(1) + p(2) + p(3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1.$$

1.2.3.2 Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF) of the discrete random variable X defined by

$$F(x) = P(X \le x) = \sum_{t \le x} p(t) = \sum_{t \le x} P(X = t). \tag{1.19}$$

Example 11: Consider the example 8, and then

1. Find : $F(-1),\ F(-0.01)\ ,F(0),\ F(0.99),\ F(1),\ F(1.99),\ F(2),\ F(2.99),\ F(3),\ F(4)$.
2. What does $F(3)$ represent?.
3. Plot $F(x)$.

Solution :

1.

$$F(-1) = \sum_{t \leq -1} p(t) = \sum_{t \leq -1} P(X = t) = 0.$$

$$F(-0.01) = \sum_{t \leq -0.01} p(t) = \sum_{t \leq -0.01} P(X = t) = 0.$$

$$F(0) = \sum_{t \leq 0} p(t) = \sum_{t \leq 0} P(X = t) = p(0) = 0.48,$$

$$F(0.99) = \sum_{t \leq 0.99} p(t) = \sum_{t \leq 0.99} P(X = t) = p(0) = 0.48,$$

$$\boxed{F(0.99) = F(0)}.$$

$$F(1) = \sum_{t \leq 1} p(t) = \sum_{t \leq 1} P(X = t) = p(0) + p(1) = 0.87,$$

$$F(1.99) = \sum_{t \leq 1.99} p(t) = \sum_{t \leq 1.99} P(X = t) = p(0) + p(1) = 0.87,$$

$$\boxed{F(1.99) = F(1)}.$$

$$F(2) = \sum_{t \leq 2} p(t) = \sum_{t \leq 2} P(X = t) = p(0) + p(1) + p(2) = 0.99,$$

$$F(2.99) = \sum_{t \leq 2.99} p(t) = \sum_{t \leq 2.99} P(X = t) = p(0) + p(1) + p(2) = 0.99,$$

$$\boxed{F(2.99) = F(2)}.$$

$$F(3) = \sum_{t \leq 3} p(t) = \sum_{t \leq 3} P(X = t) = p(0) + p(1) + p(2) + p(3) = 1,$$

$$F(4) = \sum_{t \leq 4} p(t) = \sum_{t \leq 4} P(X = t) = p(0) + p(1) + p(2) + p(3) = 1,$$

$$\boxed{F(4) = F(3)}.$$

It is also possible to write

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.48 & 0 \leq x < 1 \\ 0.87 & 1 \leq x < 2 \\ 0;99 & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

2. $F(3)$ represents the total probability That must be equal to one.
3. The Plot of $F(x)$

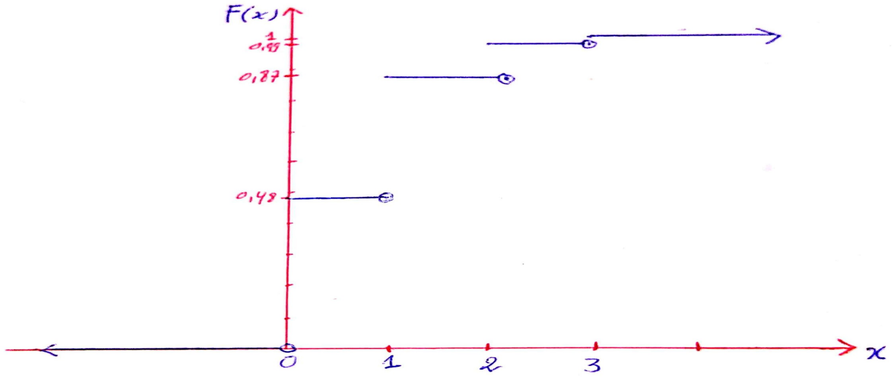

Figure 1.3: Graph of the Probability mass function of the random variable X.

1.2.3.3 Mean

The mean, which is also celled the expectation or the expected value, of a random variable X is given by

$$\langle X\rangle = \mu_X = \sum_x xP(X=x) = \sum_x xp(x). \tag{1.20}$$

The mean can also be denoted by $E(X)$ or by μ.

1.2.3.4 Moment

The nth moment is the the expected value of the X^n which is given by

$$\langle X^n\rangle = \sum_x x^n P(X=x) = \sum_x x^n p(x). \tag{1.21}$$

Note that the mean is the first moment of the random variable X.

1.2.3.5 Variance

The variance of a random variable is given by

$$\begin{aligned} Var(X) = \sigma_x^2 &= \langle (X - \langle X\rangle)^2\rangle = \sum_x (x - \langle X\rangle)^2 P(X=x) \\ &= \sum_x (x - \langle X\rangle)^2 p(x) \\ &= \sum_x \left(x^2 + \langle X\rangle^2 - 2x\langle X\rangle\right) p(x) \\ &= \sum_x x^2 p(x) + \langle X\rangle^2 \sum_x p(x) - 2\langle X\rangle \sum_x xp(x) \\ &= \langle X^2\rangle + \langle X\rangle^2 - 2\langle X\rangle^2 \\ &= \langle X^2\rangle - \langle X\rangle^2. \end{aligned} \tag{1.22}$$

The standard deviation is the square root of the variance. Therefore it is given by

$$\sigma_X = \sqrt{Var(X)} = \sqrt{\sigma_X^2} = \sqrt{\langle X^2\rangle - \langle X\rangle^2}. \tag{1.23}$$

Example 12: Consider the example 8, where the random variable has the following probability mass function

X	0	1	2	3
p(x)	0.48	0.39	0.12	0.01

1. Compute the mean of the random variable X.
2. Compute the variance then the standard deviation σ_X.

Solution :

1. The mean

$$\langle X\rangle = \mu_X = \sum_{x=0}^{3} xP(X=x) = \sum_{x=0}^{3} xp(x) = 0.48\times0+0.39\times1+0.12\times2+0.01\times3 = 0.66.$$

2. The variance

$$Var(X) = \sigma_x^2 = \langle X^2\rangle - \langle X\rangle^2,$$

$$\langle X^2\rangle = \sum_{x=0}^{3} x^2P(X=x) = \sum_{x=0}^{3} x^2p(x) = 0.48\times0+0.39\times1+0.12\times4+0.01\times9 = 0.96,$$

$$Var(X) = \sigma_x^2 = 0.96-(0.66)^2 = 0.5244.$$

Finally the standard deviation is

$$\sigma_X = \sqrt{0.5244} = 0.7241.$$

1.2.3.6 Probability Histogram

When the possible values of a discrete random variable are evenly spaced, The probability mass function can be represented by a histogram.
The area of a rectangles centered at a value x is equal to $P(X=x)=p(x)$.
Such histogram is called a probability histogram.

Example 13: Plot the histogram of example 12.

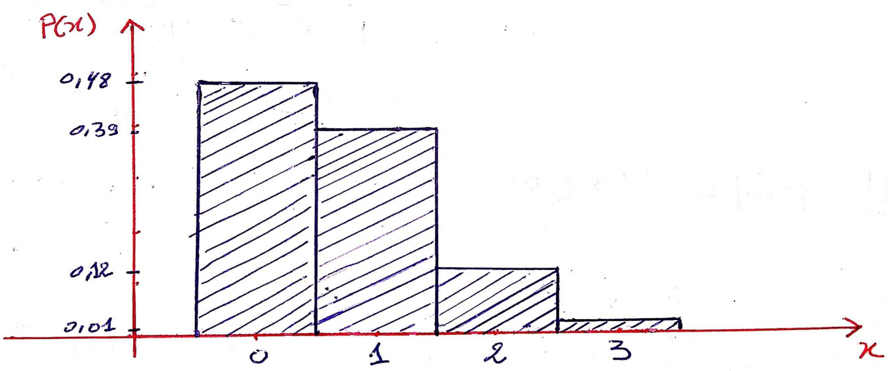

Figure 1.4: Histogram of probability mass function for the random variable X.

1.2.3.7 Some Special Distributions

A) Bernoulli Distribution

The PMF of a Bernoulli random variable with parameter p is given by

$$p(x) = P(X = x) = p^x q^{1-x} = p^x (1-p)^{1-x}, \qquad x = 0, 1, \tag{1.24}$$

where $0 \leq p \leq 1$.
The CDF of the Bernoulli random variable is then

$$F(x) = P(X \leq x) = \sum_{t \leq x} p(t) = \sum_{t \leq x} p^t q^{1-t} = \sum_{t \leq x} p^t (1-p)^{1-t}, \tag{1.25}$$

It can be also written as follows

$$F(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{1.26}$$

The mean and the variance are merely written as

$$\langle X \rangle = \mu_X = p. \tag{1.27}$$

$$Var(X) = \sigma_X^2 = pq = p(1-p). \tag{1.28}$$

A detailed derivation of 1.26 is given in the appendix A.1.
A Bernoulli random variable is associated with some experiment that can result in one of two outcomes. One of them is classified as success and its probability is p, in other word p is the probability that this event will happen in a Bernoulli trial.Then the another is classified as a failure and its probability is $q = 1-p$, in other word $q = 1-p$ is the probability that this event will fail to happen in a Bernoulli trial.

Eample 14: The random experiment of tossing a coin once is a random experiment associated with a Bernoulli random variable X.
We assume that the head is a success and the tail is a failure, thus we can write

$$X(H) = 1, \quad X(T) = 0$$

- Compute the mean and the variance of the Bernoulli random variable.

Solution :

- $$\langle X \rangle = \mu_X = \sum_{x=0}^{1} x p(x) = \sum_{x=0}^{1} x p^x (1-p)^{1-x} = 0 + 1 \times p = p.$$

- $$Var(X) = \sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2,$$

  we have to compute first $\langle X^2 \rangle$

  $$\langle X^2 \rangle = \sum_{x=0}^{1} x^2 p(x) = \sum_{x=0}^{1} x^2 p^x (1-p)^{1-x} = 0 + 1 \times p = p,$$

  $$Var(X) = p - p^2 = p(1-p).$$

B) Binomial Distribution

If a random experiment is an experiment of n independent Bernoulli trials with probability p for each success, then the random variable X represents the number of successes in the n trials is called a binomial random variable ($X \sim Bin(n,p)$). The PMF (the probability that the event will happen exactly x times in n trial) of a binomial random variable X with parameters n and p is given by

$$p(x) = P(X = x) = \begin{cases} C_{n,x}\ p^x q^{n-x} = C_{n,x}\ p^x\,(1-p)^{n-x} & x = 0, 1, 2, \cdots, n \\ 0 & otherwise \end{cases} \tag{1.29}$$

where $0 \le p < 1$ and the binomial coefficient is given by

$$C_{n,x} = \binom{n}{x} = \frac{n!}{x!\,(n-x)!} \tag{1.30}$$

The CDF is then given by

$$F(x) = P(X \le x) = \sum_{t \le x} p(t) = \sum_{t \le x} C_{n,x}\ p^t q^{n-t} = \sum_{t \le x} C_{n,x}\ p^t (1-p)^{n-t}, \tag{1.31}$$

The mean and the variance of a binomial random variable X are given by

$$\langle X \rangle = \mu_X = np. \tag{1.32}$$

$$Var(X) = \sigma_X^2 = npq = np(1-p). \tag{1.33}$$

The proof of 1.32 and 1.33 is done in the appendix A.2.

C) Poisson Distribution

A random variable X is a Poisson random variable with parameter $\lambda(\lambda > 0)$ if its PMF takes the form

$$p(x) = P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}, \qquad x = 0, 1, 2, 3, \cdots. \tag{1.34}$$

The CDF of the Poisson random variable is

$$F(x) = P(X \le x) = e^{-\lambda}\sum_{t \le x}\frac{\lambda^t}{t!}. \tag{1.35}$$

The mean and the variance of the Poisson random variable are as follows

$$\langle X \rangle = \mu_X = \lambda. \tag{1.36}$$

$$Var(X) = \sigma_X^2 = \lambda \tag{1.37}$$

D) Geometric Distribution

A random variable X is a geometric random variable with parameter p if its PMF takes the form

$$p(x) = P(X = x) = pq^{x-1} = p\,(1-p)^{x-1}, \qquad x = 1, 2, 3, \cdots. \tag{1.38}$$

The CDF of the Poisson random variable is

$$
\begin{aligned}
F(x) &= P(X \leq x) = \sum_{t \leq x} pq^{t-1} \\
&= \sum_{t=1}^{x} pq^{t-1} = p\frac{1-q^x}{1-q}.
\end{aligned} \tag{1.39}
$$

Since $q = 1 - p$ then 1.39 becomes

$$
F(x) = p\frac{1-(1-p)^x}{1-(1-p)} = 1-(1-p)^x. \tag{1.40}
$$

The mean and the variance of the Poisson random variable are as follows

$$
\langle X \rangle = \mu_X = \frac{1}{p} \tag{1.41}
$$

$$
Var(X) = \sigma_X^2 = \frac{q}{p^2} = \frac{1-p}{p^2} \tag{1.42}
$$

Remark The geometric random variable X denotes the number of Bernoulli trials on which Bernoulli event happens at the first time (the first success occurs).

B) Negative Binomial Distribution

A random variable X that represents the number of independent Bernoulli trials until the kth event happens (kth success occurs) is called a negative binomial random variable. The PMF of a negative binomial random variable X with parameters p and k is given by

$$
p(x) = P(X = x) = \begin{cases} C_{x-1,k-1}\; p^k q^{x-k} = C_{x-1,k-1}\; p^k (1-p)^{x-k} & x = k, k+1, k+2, \cdots \\ 0 & otherwise \end{cases} \tag{1.43}
$$

where $0 \leq p < 1$ and the binomial coefficient is given by

$$
C_{x-1,k-1} = \binom{x-1}{k-1} = \frac{(x-1)!}{(k-1)!\,(x-k)!} \tag{1.44}
$$

The CDF is then given by

$$
F(x) = P(X \leq x) = \sum_{t \leq x} p(t) = \sum_{t \leq x} C_{t-1,k-1}\; p^k q^{t-k} = \sum_{t \leq x} C_{t-1,k-1}\; p^k (1-p)^{t-k}, \tag{1.45}
$$

The mean and the variance of a binomial random variable X are given by

$$
\langle X \rangle = \mu_X = \frac{k}{p}. \tag{1.46}
$$

$$
Var(X) = \sigma_X^2 = \frac{k\;q}{p} = \frac{k\;(1-p)}{p}. \tag{1.47}
$$

Remarks

- It clear to note that when $k = 1$, X is exactly a geometric random variable.
- A negative binomial random variable is sometimes called a Pascal random variable.

D) Discrete Uniform Distribution

A random variable X is a discrete uniform random variable if its PMF takes the form

$$p(x) = P(X = x) = \frac{1}{n}, \qquad 1 \leq x \leq n. \tag{1.48}$$

The CDF of the discrete uniform random variable is

$$F(x) \;=\; P(X \leq x) = \sum_{t \leq x} \frac{1}{n}, \tag{1.49}$$

It can be written as follows

$$F(x) = \begin{cases} 0 & x < 0 < 1 \\ \frac{\lfloor x \rfloor}{n} & 1 < x < n \\ 1 & x \geq n \end{cases} \tag{1.50}$$

The mean and the variance of the Poisson random variable are as follows

$$\langle X \rangle = \mu_X = \frac{1}{2}(n + 1). \tag{1.51}$$

$$Var(X) = \sigma_X^2 = \frac{1}{12}(n^2 - 1). \tag{1.52}$$

Remarks

- It is not possible to define a discrete uniform distribution for a random variable X if its sample space is not countably finite set.

1.2.4 Continuous Random Variable

A random variable is said a continuous random variable if its possible values form an interval (either finite or infinite) of real numbers.

1.2.3.1 Probability Mass Function

Appendix A

Some special Distribution Details

A.1 Bernoulli Distribution

The detailed calculation to derive the 1.26, is given as follows

$$F(-0.01) = \sum_{t\leq -0.01} p^t(1-p)^{1-t} = 0.$$
$$F(0) = \sum_{t\leq 0} p^t(1-p)^{1-t} = 1-p.$$
$$F(1) = \sum_{t\leq 1} p^t(1-p)^{1-t} = (1-p)+p = 1.$$

Thus the CDF takes the form that has written above 1.26.

A.2 Binomial Distribution

The proof of 1.32

$$\begin{aligned}
\langle X\rangle &= \sum_{x=0}^{n} xC_{n,x}\ p^x\,(1-p)^{n-x} \\
&= \sum_{x=0}^{n} x\frac{n!}{x!\,(n-x)!}p^x\,(1-p)^{n-x} \\
&= 0+\sum_{x=1}^{n} x\frac{n!}{x!\,(n-x)!}p^x\,(1-p)^{n-x} \\
&= \sum_{x=1}^{n} x\frac{n(n-1)!}{x(x-1)!\,(n-1-(x-1))!}p\ p^{x-1}\,(1-p)^{n-1-(x-1)} \\
&= np\sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-1-(x-1))!}p^{x-1}\,(1-p)^{n-1-(x-1)}, \qquad \text{(A.1)}
\end{aligned}$$

Let's make the following changes

$$y = x-1,\ n' = n-1 \longrightarrow \begin{cases} x=1, & y=0 \\ x=n, & y=n-1=n' \end{cases} \qquad \text{(A.2)}$$

by using this later changes, we get

$$\begin{aligned}\langle X\rangle &= np\sum_{y=0}^{n-1}\frac{(n-1)!}{y!\,(n-1-y)!}p^{y}\,(1-p)^{n-1-y}\\ &= np\sum_{y=0}^{n'}\frac{n'!}{y!\,(n'-y)!}p^{y}\,(1-p)^{n'-y}\\ &= np\sum_{y=0}^{n'}\binom{n'}{y}p^{y}\,(1-p)^{n'-y}\,. \end{aligned}\tag{A.3}$$

The binomial formula or the binomial identity is written as

$$(A+B)^{n} = \sum_{x=0}^{n}\binom{n}{x}A^{x}B^{n-x}. \tag{A.4}$$

Finally, using the binomial formula A.4, we get

$$\langle X\rangle = np\,(p+(1-p))^{n'} = np. \tag{A.5}$$

We can also use another method to prove the result that we have found above 1.32

$$f(p)=(p+q)^{n} = \sum_{x=0}^{n}C_{n,x}\ p^{x}\ q^{n-x}. \tag{A.6}$$

We derive the function $f(p)$ and then we multiply the sides of the identity by p, we get

$$pf'(p)=np(p+q)^{n-1} = \sum_{x=0}^{n}xC_{n,x}\ p^{x}\ q^{n-x}=\langle X\rangle. \tag{A.7}$$

Since the derived formula A.7 is held valid when we take q to be equal to $1-p$, then

$$\langle X\rangle = np\,(p+(1-p))^{n-1} = np. \tag{A.8}$$

The proof of 1.33

First we have to calculate $\langle X^2\rangle$.

Now, by computing the second derivative of $f(p)$ and by multiply the sides of the identity by p^2, we get

$$\begin{aligned}p^{2}f''(p) &= n(n-1)p^{2}(p+q)^{n-2}=\sum_{x=0}^{n}x(x-1)\ C_{n,x}\ p^{2}p^{x-2}\ q^{n-x}\\ &= \sum_{x=0}^{n}x^{2}\ C_{n,x}\ p^{x}\ q^{n-x}-\sum_{x=0}^{n}x\ C_{n,x}\ p^{x}\ q^{n-x}\\ &= \langle X^{2}\rangle-\langle X\rangle, \end{aligned}\tag{A.9}$$

After a simple arrangement and by taking $q=1-p$, we get

$$\begin{aligned}\langle X^{2}\rangle &= p^{2}f''(p)+\langle X\rangle = n(n-1)p^{2}(p+q)^{n-2}-np\\ &= n(n-1)p^{2}(p+(1-p))^{n-2}-np=n^{2}p^{2}+np(1-p). \end{aligned}\tag{A.10}$$

Eventually, The variance can merely computed as follows

$$Var(X)=\sigma(X)^{2}=\langle X^{2}\rangle-\langle X\rangle^{2}=n^{2}p^{2}+np(1-p)-n^{2}p^{2}=np(1-p). \tag{A.11}$$

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