1-2-4-1- Probability Density function:

A Continuous random variable is associated with a probability density function $p(x)$.

The probability density function is sometimes called the probability distribution.

Let a and b be any two numbers, with $a < b$, then

1) $P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b)$

$$= P(a < x < b) = \int_a^b p(x)\,dx .$$

2) $P(X \leq b) = P(X < b) = \int_{-\infty}^{b} p(x)\,dx$

3) $P(X \geq a) = P(X > a) = \int_a^{+\infty} p(x)\,dx .$

Remark: If $a = -\infty$ and $b = +\infty$, then

$$\int_{-\infty}^{+\infty} p(x)\,dx = 1 .$$

This result represents the total probability.

Example: The probability density function of a random variable X given by:

$$p(x) = \begin{cases} 1{,}25(1-x^4) & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- Calculate the probability $P(X \geq 0.8)$ then $P(0 < X < 1)$

Solution:

$$P(X > 0.8) = \int_{0.8}^{+\infty} p(x)\,dx = \int_{0.8}^{1} 1.25(1-x^4)\,dx$$

$$= 1.25\left(x - \frac{x^5}{5}\right)\Big|_{0.8}^{1} = 0.0819$$

$$P(0 < X < 1) = \int_{0}^{1} 1.25(1-x^4)\,dx.$$

$$= \frac{5}{4}\left(x - \frac{x^5}{5}\right)\Big|_{0}^{1}$$

$$= \frac{5}{4}\left(1 - \frac{1}{5}\right) = \frac{5}{4} \times \frac{4}{5} = 1.$$

1-2-4-2- Cumulative Distribution Function:

Let X be a Continuous random variable with probability density function $p(x)$.

The cumulative distribution function of X is the function:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} p(t)\,dt.$$

Exemple: Find the CDF of X in the previous example if $x = 0.5$, $x = 0$.

Solution:

$$F(0.5) = P(X \leq 0.5) = P(X < 0.5)$$

$$F(0) = P(X \leq 0) = P(X < 0) = \int_{-\infty}^{0} p(t)\,dt$$

$$= \int_{-\infty}^{0} 0\,dt = 0$$

$$F(0{,}5) = P(X \leq 0{,}5) = P(X < 0{,}5) = \int_{-\infty}^{0{,}5} p(t)\,dt$$

$$= \cancel{\int_{-\infty}^{0} p(t)\,dt}^{\,0} + \int_{0}^{0{,}5} p(t)\,dt$$

$$= \frac{5}{4} \int_{0}^{0{,}5} (1 - t^4)\,dt = \frac{5}{4}\left(t - \frac{t^5}{5}\right)\Bigg|_{0}^{0{,}5}$$

$$=$$

In general:

$$F(x) = \int_{-\infty}^{x} p(t)\,dt = \cancel{\int_{-\infty}^{0} p(t)\,dt}^{\,0} + \int_{0}^{x} p(t)\,dt$$

$$= \frac{5}{4} \int_{0}^{x} (1 - t^4)\,dt$$

$$= \frac{5}{4}\left(t - \frac{t^5}{5}\right)\Bigg|_{0}^{x}$$

$$= \frac{5}{4}\left(x - \frac{x^5}{5}\right).$$

1-2-4-3- Mean:

The mean of the continuous random variable X is given by:

$$\langle X \rangle = \mu_X = \int_{-\infty}^{+\infty} x\, P(X \leq x)\, dx = \int_{-\infty}^{+\infty} x\, p(x)\, dx .$$

$\langle X \rangle$ or μ_X may also be denoted by $E(X)$ or even by μ.

1-2-4-4- Moment:

The n-th moment of a random variable X is

$$\langle X^n \rangle = \int_{-\infty}^{+\infty} x^n\, P(X \leq x)\, dx$$

$$= \int_{-\infty}^{+\infty} x^n\, p(x)\, dx .$$

1-2-4-5- The Variance:

The variance of a continuous random variable is given by

$$Var(X) = \langle X^2 \rangle - \langle X \rangle^2 = \int_{-\infty}^{+\infty} (x - \langle X \rangle)^2\, p(x)\, dx$$

An alternative formula for the variance may takes the form:

$$Var(X) = \int_{-\infty}^{+\infty} x^2\, p(x)\, dx - \mu_X^2$$

The standard deviation is then:

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{Var(X)} = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$$

1-2-4-6- Some Special distribution:

- Uniform Distribution:

A random variable is called a continuous uniform random variable if its pdf is given by:

$$P(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

- Exponential Distribution;

A r.v X is called an exponential r.v with parameter λ (>0) if its pdf is given by:

$$P(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x < 0. \end{cases}$$

- Gamma Distribution:

A r.v is called a gamma r.v with parameters (α, λ) ($\alpha > 0$ and $\lambda > 0$) if its pdf is given by:

$$P(x) = \begin{cases} \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x > 0 \\ 0 & x < 0 \end{cases}$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx.$

- Normal (Gaussian) Distribution:

A r.v is called a normal (Gaussian) r.v if its pdf is given by:

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

1-3- Independence and Simple Random Samples:

- If $X_1, X_2, \ldots, X_n$ is a simple random sample. Then $X_1, X_2, \ldots, X_n$ may be treated as independent random variables, all with the same distribution..

- When $X_1, \ldots, X_n$ are independent random variables, all with the same distribution, it is sometimes said that $X_1, \ldots, X_n$ are independent and identically distributed.

1-3-1- The mean and Variance of a

Sample mean:

The most frequently encountered linear combination is the sample mean. Specifically if $X_1, \ldots, X_n$ is a simple random sample from a population with mean μ and variance σ^2, then the sample mean $\overline{X}$ is the linear combination

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

From this fact we compute the mean and variance of $\overline{X}$:

$$\mu_{\overline{X}} = \langle \overline{X} \rangle = \frac{1}{n} \langle X_1 + \cdots + X_n \rangle$$

$$= \frac{1}{n} \langle X_1 \rangle + \cdots + \frac{1}{n} \langle X_n \rangle$$

$$= \frac{1}{n} \mu_{X_1} \cdots + \frac{1}{n} \mu_{X_n}$$

$$= \frac{1}{n} \mu + \cdots + \frac{1}{n} \mu$$

$$= \frac{1}{n} \mu \times n = \mu = \mu_X$$

$$Var(\overline{X}) = \langle \overline{X}^2 \rangle - \langle \overline{X} \rangle^2$$

$$= \frac{1}{n^2} \langle X_1^2 + \cdots + X_n^2 \rangle - \frac{1}{n^2} \left[\langle X_1 \rangle^2 + \cdots + \langle X_n \rangle^2 \right]$$

$$= \frac{1}{n^2} \left[\langle X_1^2 \rangle - \langle X_1 \rangle^2 \right] + \cdots + \frac{1}{n^2} \left[\langle X_n^2 \rangle - \langle X_n \rangle^2 \right]$$

$$= \frac{1}{n^2} \sigma_{X_1}^2 + \cdots + \frac{1}{n^2} \sigma_{X_n}^2$$

$$= n \left(\frac{1}{n^2} \sigma^2 \right)$$

$$= \frac{\sigma^2}{n}.$$

The standard deviation of $\overline{X}$ is

$$\sigma_{\overline{X}} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma^2}{\sqrt{n}}.$$