and consequently we have $x \in \overline{\bigcap_{n \geq 0} D(A^n)}$. The proof is complete. □

We conclude this section with both a consequence and a completion of (iii) in Theorem 2.3.2.

Corollary 2.4.1. *Let $n \in \mathbb{N}^*$. Then, for each $\xi \in D(A^n)$ and each $t \geq 0$, we have $S(t)\xi \in D(A^n)$, the function $u : [0, +\infty) \to X$, $u(t) = S(t)\xi$ is of class C^n and is a solution of the Cauchy problem*

$$\begin{cases} u^{(n)}(t) = A^n u(t), & t \geq 0 \\ u^{(k)}(0) = A^k \xi, & k = 0, 1, \ldots, n-1. \end{cases}$$

Proof. We proceed by mathematical induction. Let us remark that for $n = 1$ the conclusion follows from (iii) in Theorem 2.3.2. Let us assume that the property in question holds for $n \in \mathbb{N}^*$ and let $\xi \in D(A^{n+1})$. Since $D(A^{n+1}) \subseteq D(A^n)$, the inductive hypothesis yields $u^{(n)}(t) = A^n u(t)$ for each $t \geq 0$. Let $t \geq 0$ and $h \in \mathbb{R}$ with $t + h \geq 0$. We have

$$\frac{1}{h}\left(u^{(n)}(t+h) - u^{(n)}(t)\right) = \frac{1}{h}\left(A^n S(t+h)\xi - A^n S(t)\xi\right)$$
$$= \frac{1}{h}\left(S(t+h)A^n\xi - S(t)A^n\xi\right)$$

because $\xi \in D(A^n)$, while, for each $\tau \geq 0$, A^n and $S(\tau)$ commutes on $D(A^n)$. But $A^n\xi \in D(A)$, and consequently there exists

$$\lim_{h \downarrow 0} \frac{1}{h}\left(S(t+h)A^n\xi - S(t)A^n\xi\right) = AS(t)A^n\xi.$$

Passing to the limit for $h \to 0$ in the preceding equality, and taking into account that $A^n\xi \in D(A)$, we deduce

$$u^{(n+1)}(t) = AS(t)A^n\xi = A^{(n+1)}S(t)\xi.$$

Clearly $u^k(0) = A^k\xi$ for $k = 0, 1, 2, \ldots n$ and this completes the proof. □

Problems

Let $p \in [1, +\infty)$ and let $X = l_p$ be the space of real sequences $(x_n)_{n \in \mathbb{N}^*}$ satisfying $\sum_{n=1}^{\infty} |x_n|^p < +\infty$. This space, endowed with the norm $\|\cdot\|_p$, defined by $\|(x_n)_{n\in\mathbb{N}}\|_p = \left(\sum_{n=1}^{\infty}|x_n|^p\right)^{1/p}$ for each $(x_n)_{n\in\mathbb{N}} \in l_p$, is a real Banach space.

Problem 2.1. Let $p \in [1, +\infty)$, $X = l_p$, $(a_n)_{n\in\mathbb{N}^*}$ a sequence of positive real numbers, and $t \in \mathbb{R}_+$. We define $S(t) : D(A) \subseteq X \to X$ by

$$(S(t)(x_n)_{n\in\mathbb{N}^*})_{k\in\mathbb{N}^*} = \left(e^{-a_k t}x_k\right)_{k\in\mathbb{N}^*}$$

for each $(x_n)_{n\in\mathbb{N}^*} \in l_p$.

(i) Prove that $\{S(t)\,;\ t \geq 0\}$ is a C_0-semigroup of contractions on l_p.
(ii) Find its infinitesimal generator.
(iii) Prove that this semigroup is uniformly continuous if and only if $(a_n)_{n\in\mathbb{N}^*}$ is bounded.

Let c_0 be the space of real sequences vanishing at ∞. Endowed with the norm $\|\cdot\|_\infty$, defined by $\|(x_n)_{n\in\mathbb{N}^*}\|_\infty = \sup_{n\in\mathbb{N}^*}|x_n|$ for each $(x_n)_{n\in\mathbb{N}^*} \in c_0$, this is real Banach space.

Problem 2.2. Let $X = c_0$, $(a_n)_{n\in\mathbb{N}^*}$ a sequence of positive real numbers and $t \in \mathbb{R}_+$. Let us define $S(t) : D(A) \subseteq X \to X$ by

$$(S(t)(x_n)_{n\in\mathbb{N}^*})_{k\in\mathbb{N}^*} = \left(e^{-a_k t}x_k\right)_{k\in\mathbb{N}^*}$$

for each $(x_n)_{n\in\mathbb{N}^*} \in c_0$.

(i) Prove that $\{S(t)\,;\ t \geq 0\}$ is a C_0-semigroup of contractions.
(ii) Find its infinitesimal generator.
(iii) Prove that this semigroup is uniformly continuous if and only if $(a_n)_{n\in\mathbb{N}^*}$ is bounded.

Problem 2.3. Let $X = C_b(\mathbb{R})$ (the space of all continuous and bounded functions from $\mathbb{R}$ to $\mathbb{R}$, which is a real Banach space with respect to the sup-norm), let $t \in \mathbb{R}_+$, and let $S(t) : X \to X$ defined by $[S(t)f](s) = f(t+s)$ for each $f \in X$, and each $s \in \mathbb{R}$. Show that $\{S(t)\,;\ t \geq 0\}$ is a semigroup of linear operators, which is not of class C_0. Find its infinitesimal generator and show that $D(A)$ is not dense in X.

Problem 2.4. Let $X = C_{ub}(\mathbb{R})$ endowed with the norm supremum, let $t \in \mathbb{R}$, $\lambda > 0$, $\delta > 0$ and let us define $G(t) : X \to X$ by

$$[G(t)f](x) = e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}f(x - k\delta)$$

for each $f \in X$ and each $x \in \mathbb{R}$. Prove that $\{G(t)\,;\ t \in \mathbb{R}\}$ is a uniformly continuous group of isometries, whose infinitesimal generator, $A : X \to X$, is defined by $[Af](x) = \lambda[f(x-\delta) - f(x)]$ for each $f \in X$, and each $x \in \mathbb{R}$. This is Exercise 9, p. 23 in Goldstein [**61**].

Problem 2.5. Let $X = L^p(\mathbb{R}^n)$, let $\mathcal{A}$ be an $n \times n$ matrix with real entries and let us define $G(t) : X \to X$ by $[G(t)f](x) = f\left(e^{-t\mathcal{A}}x\right)$ for each $t \in \mathbb{R}$, $f \in X$ and a.e. for $x \in \mathbb{R}^n$. Prove that $\{G(t)\,;\ t \in \mathbb{R}\}$ is a C_0-group and find its infinitesimal generator. Show that, if $\sum_{i=1}^{n} a_{ii} = 0$, then the group is of isometries.

Problem 2.6. Show that, with X replaced by $C_{ub}(\mathbb{R}^n)$, i.e. the space of uniformly continuous and bounded functions from $\mathbb{R}^n$ to $\mathbb{R}$, endowed with the sup-norm, the family $\{G(t)\,;\ t \in \mathbb{R}\}$ defined as in Problem 2.5, although a group, is not a C_0-group.

Problem 2.7. Let $\{S(t)\,;\ t \geq 0\}$ be a semigroup of linear operators with the property that, for each $x \in X$, we have $\lim_{t\downarrow 0} S(t)x = x$ in the weak topology of X. Prove that there exists $M \geq 1$, and $\omega \in \mathbb{R}$, such that $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{t\omega}$ for each $t \geq 0$.

Problem 2.8. Let $\{S(t)\,;\ t \geq 0\}$ be a semigroup of linear operators with the property that, for each $x \in X$, we have $\lim_{t\downarrow 0} S(t)x = x$ in the weak topology of X. Prove that $\{S(t)\,;\ t \geq 0\}$ is a C_0-semigroup. This is Dunford theorem. See Pazy [**101**], Theorem 1.4, p. 44, or Engel and Nagel et al, Theorem 5.8, p. 40.

Notes. The main results in Sections 2.1 and 2.2, referring to uniformly continuous groups, were obtained independently by Nathan [**93**], Nagumo [**92**] and by Yosida [**134**], but they have their roots in the pioneering works of Peano [**102**], [**103**] concerning the exponential function of a matrix. Sections 2.3 and 2.4 contain several classical notions and results, which may be found, in one form or another in the monographs and treatises on semigroup theory mentioned in Preface. With some exceptions, the problems included are adapted from Brezis and Cazenave [**31**], Engel and Nagel et al [**51**], Goldstein [**61**] and Pazy [**101**].