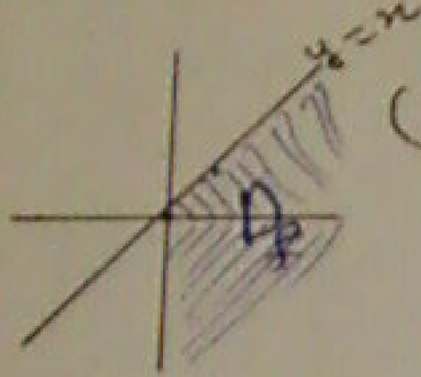
حل السلسلة 5 (الدوال العددية لمتغيرين)

تمرين ①:

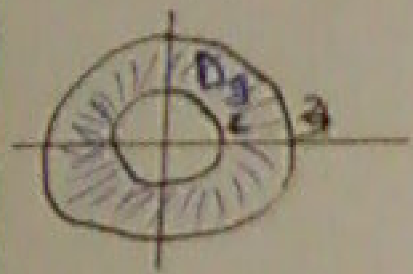


① $(f \text{ معرفة}) \Leftrightarrow (x \geq 0 \wedge x-y \geq 0)$

$\Leftrightarrow (x \geq 0 \wedge y \leq x)$

$D_f = \{(x,y) \in \mathbb{R}^2 / \ x \geq 0 \wedge y \leq x\}$

② $(g \text{ معرفة}) \Leftrightarrow (x^2+y^2-4>0 \wedge 9-x^2-y^2>0)$



$\Leftrightarrow (4 < x^2+y^2 < 9)$

$D_g = \{(x,y) \in \mathbb{R}^2 / \ 4 < x^2+y^2 < 9\}$

(أو $2 < \sqrt{x^2+y^2} < 3$)

تمرين ②:

1) استمرارية f:
• f مستمرة على $\mathbb{R}^2 \setminus \{(0,0)\}$

• عند $(0,0)$: بوضع $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{\cancel{r^2}\cos\theta\sin\theta(\cos^2\theta-\sin^2\theta)r^2}{\cancel{r^2}} = 0$$

b ①: حساب النهاية وفق السبيل $y=mx$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{mx^2(x^2-m^2x^2)}{x^2+m^2x^2} = \lim_{x\to 0} \frac{m\cancel{x^2}(1-m^2)x^2}{\cancel{x^2}(1+m^2)} = 0$$

وبالتالي f مستمرة عند $(0,0)$

إذن f مستمرة على $\mathbb{R}^2$.

(2

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim \frac{0-0}{h} = 0$$

ولدينا:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0} \frac{h - 0}{h} = 1$$ ولدينا:

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0} \frac{-h - 0}{h} = -1$$

ولدينا:

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \begin{cases} \dfrac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \begin{cases} \dfrac{x^6 + 13x^4y^2 + 7x^2y^4 - 16x^3y^2 - 4x^5y + 4xy^5 - 5y^6}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ -1, & (x,y) = (0,0) \end{cases}$$

ت 3 :

1) ط 1

$$I_1 = \int_1^2 \int_0^2 (x+y)\, e^{x+y}\, dx\, dy = \int_1^2\int_0^2 (xe^x e^y + y e^y e^x)\, dx\, dy$$

$$= \int_1^2 \left[\int_0^2 (e^y x e^x + y e^y e^x)\, dx \right] dy$$

$$= \int_1^2 \left(e^y \left[xe^x - e^x \right]_0^2 + y e^y \left[e^x \right]_0^2 \right) dy$$

$$= (e^2+1)\left[e^y\right]_1^2 + (e^2-1)\left[ye^y - e^y\right]_1^2$$

$$= (e^2+1)(e^2-e) + (e^2-1)(e^2) = \boxed{2e^4 - e^3 - e}$$

ط 1

$$I_1 = \int_0^2\int_1^2 (x+y)\, e^{x+y}\, dy\, dx = \int_0^2\int_1^2 (xe^x e^y + e^x y e^y)\, dy\, dx$$

$$= \int_0^2 \left[\int_1^2 (xe^x e^y + e^x y e^y)\, dy \right] dx$$

$$= \int_0^2 \left(xe^x \left[e^y\right]_1^2 + e^x \left[ye^y - e^y\right]_1^2 \right) dx$$

$$= (e^2 - e)\left[xe^x - e^x\right]_0^2 + (e^2)\left[e^x\right]_0^2$$

$$= (e^2-e)(e^2+1) + e^2(e^2-1) = \boxed{2e^4 - e^3 - e}$$

2) نستعمل الإحداثيات القطبية :

نضع $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ لنجد $dx\,dy = r\,dr\,d\theta$ ويصبح

$$I_2 = \iint_D r^2\,dr\,d\theta$$

من $x^2+y^2-1 \leq 0$ نجد $r^2 \leq 1$ أي $r \leq 1$

ومن $x^2+y^2-2y \geq 0$ نجد $r^2 \geq 2r\sin\theta$ أي $r \geq 2\sin\theta$

ومنه $2\sin\theta \leq r \leq 1$

ومن $x \geq 0$, $y \geq 0$ فإن $\cos\theta \geq 0$, $\sin\theta \geq 0$ ولكون $\sin\theta \leq \frac{1}{2}$ ينتج $0 \leq \theta \leq \frac{\pi}{6}$

وبالتالي :

$$I_2 = \int_0^{\frac{\pi}{6}}\left[\int_{2\sin\theta}^{1} r^2\,dr\right]d\theta = \frac{1}{3}\int_0^{\frac{\pi}{6}}\left[r^3\right]_{2\sin\theta}^{1}d\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{6}}(1-8\sin^3\theta)\,d\theta$$

نحسب

$$\int_0^{\frac{\pi}{6}}\sin^3\theta\,d\theta = \int_0^{\frac{\pi}{6}}(1-\cos^2\theta)\sin\theta\,d\theta$$

نضع $t = \cos\theta$ لنجد $dt = -\sin\theta\,d\theta$

ومن $\theta = 0$ نجد $t = 1$

ومن $\theta = \frac{\pi}{6}$ نجد $t = \frac{\sqrt{3}}{2}$

إذن

$$\int_0^{\frac{\pi}{6}}\sin^3\theta\,d\theta = -\int_1^{\frac{\sqrt{3}}{2}}(1-t^2)\,dt = \left[\frac{t^3}{3}-t\right]_1^{\frac{\sqrt{3}}{2}} = -\frac{3\sqrt{3}}{8}+\frac{2}{3}$$

وبالتالي

$$I_2 = \frac{1}{3}\left[\left[t\right]_0^{\frac{\pi}{6}} - 8\left(-\frac{3\sqrt{3}}{8}+\frac{2}{3}\right)\right] = \frac{1}{3}\left(\frac{\pi}{6}+3\sqrt{3}-\frac{16}{3}\right)$$

$$= \frac{\pi}{18}+\sqrt{3}-\frac{16}{9}$$

3) نضع $u = x-y$ و $v = x+y$ لنجد $x = \frac{u+v}{2}$ و $y = \frac{-u+v}{2}$

لدينا : $dx\,dy = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} du\,dv$ ، $\left(dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv\right)$

أي $dx\,dy = \frac{1}{2}\,du\,dv$

من $x \geq 0$ و $y \geq 0$ فإن $u+v \geq 0$ و $-u+v \geq 0$ ومنه $-v \leq u \leq v$

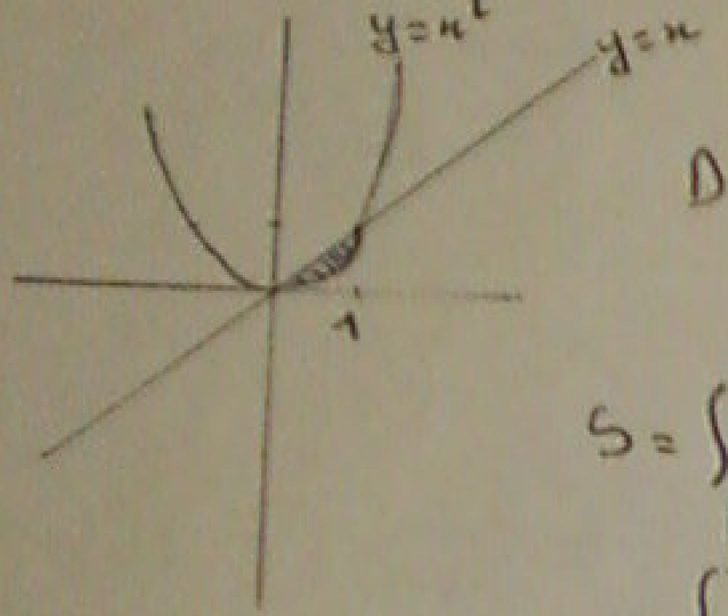
ومن $0 \leq x+y \leq 1$ فإن $0 \leq v \leq 1$

و بالتالي :

$$I_3 = \iint_D e^{\frac{x-y}{x+y}} dx\,dy = \int_0^1 \int_{-v}^{v} e^{\frac{u}{v}}\, du\, dv$$

$$= \int_0^1 \left[\int_{-v}^{v} e^{\frac{1}{v}u}\, du \right] dv = \int_0^1 v \left[e^{\frac{1}{v}u} \right]_{-v}^{v} dv = (e - e^{-1}) \int_0^1 v\, dv$$

$$= \left(e - \frac{1}{e}\right) \left[\frac{v^2}{2} \right]_0^1 = \frac{1}{2}\left(e - \frac{1}{e}\right)$$



ت④ : المساحة D معرّفة كمايلي :

$$D = \{ (x, y) \in \mathbb{R}^2 \ / \ 0 \leq x \leq 1 \ \wedge \ x^2 \leq y \leq x \}$$

فالمساحة المطلوبة :

$$S = \iint_D dx\,dy = \int_0^1 \int_{x^2}^{x} dy\, dx$$

$$= \int_0^1 \left[\int_{x^2}^{x} dy \right] dx = \int_0^1 [y]_{x^2}^{x}\, dx$$

$$= \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6} \text{ و.م}}$$