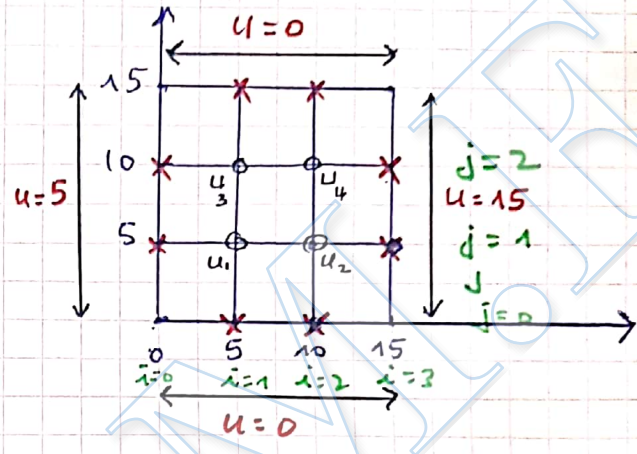
Exemple: $\nabla^2 u = 0$; $u(x,0) = 0$ ; $u(0,y) = 5$

$u(x,15) = 0$ ; $u(15,y) = 15$ $\quad h = 5$

Solution:

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = h^2 . f(x,y).$$

$u_{11} = u_1$ at $i=1$ ; $j=1$




$$u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 4u_{1,1} = 0$$

$$u_{2,1} + 5 + u_{1,2} + 0 - 4u_{11} = 0$$

$$u_2 + 5 + u_3 + 0 - 4u_1 = 0$$

$u_2$ at $i=2$ ; $j=1$

$$u_1 + 15 + u_4 + 0 - 4u_2 = 0$$

$u_3$ at $i=1$ , $j=2$

$$5 + u_4 + 0 + u_1 - 4u_3 = 0$$

$u_4$ at $i=2$ , $j=2$

$$15 + u_3 + 0 + u_2 - 4u_4 = 0$$

$$\begin{cases} -4u_1 + u_2 + u_3 = -5 \\ u_1 - 4u_2 + u_4 = -15 \\ u_1 + u_4 - 4u_3 = -5 \\ u_2 + u_3 - 4u_4 = -15 \end{cases} \qquad \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -5 \\ -15 \\ -5 \\ -15 \end{bmatrix}$$

- $u_1 = 3.5 \quad u_2 = 5.7 \quad u_3 = 3.2 \quad u_4 = 4.56$

- méthode de Solution:

- $u_1 = \dfrac{u_2 + u_3}{4} + \dfrac{5}{4}$ • $u_3 = \dfrac{u_1 + u_4}{4} + \dfrac{5}{4}$
- $u_2 = \dfrac{u_1 + u_4}{4} + \dfrac{15}{4}$ • $u_4 = \dfrac{u_2 + u_3}{4} + \dfrac{15}{4}$

initial gausse: $(u_1^{(0)}, u_2^{(0)}, u_3^{(0)}, u_4^{(0)}) = (0, 0, 0, 0)$,

G-Jacobi : Gausse-Jacobi

$u_1^{(1)} = 5/4 = 1,25$ ; $u_2^{(1)} = 15/4 = 3.75$ ; $u_3^{(1)} : 5/4 = 1,25$ ; $u_4^{(1)} = 3.75$

$$u_1^{(2)} = \frac{u_2^{(1)} + u_3^{(1)}}{4} + \frac{5}{4} = \frac{3.75 + 1.25}{4} + \frac{5}{4} = 2,5$$

$$u_2^{(2)} = \frac{u_1^{(1)} + u_4^{(1)}}{4} + \frac{15}{4} = \frac{1.25 + 3.75}{4} + \frac{15}{4} = 5$$

$$u_3^{(2)} = \frac{u_1^{(1)} + u_4^{(1)}}{4} + \frac{5}{4} = \frac{5}{4} + \frac{5}{4} = 2,5$$

$$u_4^{(2)} = \frac{u_2^{(1)} + u_3^{(1)}}{4} + \frac{15}{4} = \frac{5}{4} + \frac{15}{4} = 5$$

$$\begin{cases} u_1 = \frac{u_2+u_3}{4} + \frac{5}{4} \\ u_2 = \frac{u_1+u_4}{4} + \frac{15}{4} \\ u_3 = \frac{u_1+u_4}{4} + \frac{5}{4} \\ u_4 = \frac{u_2+u_3}{4} + \frac{15}{4} \end{cases}$$

Gauss Seidel: $(u_1^{(0)}, u_2^{(0)}, u_3^{(0)}, u_4^{(0)}) = (0,0,0,0)$

$$u_1^{(1)} = 5/4 = 1.25$$

$$u_2^{(1)} = \frac{u_1^{(1)} + u_4^{(0)}}{4} + \frac{15}{4} = \frac{15+1,25}{4} = 4.06$$

$$u_3^{(1)} = \frac{u_1^{(1)} + u_4^{(0)}}{4} + \frac{5}{4} = \frac{1,25}{4} + \frac{15}{16} + \frac{5}{4} = \frac{5+15+20}{16} = 2,5$$

$$u_4^{(1)} = \frac{u_2^{(1)} + u_3^{(1)}}{4} + \frac{15}{4} = \frac{4.06 + 2,5 + 15}{4} = 5.39$$

$$u_1^{(2)} = \frac{u_2^{(1)} + u_3^{(1)}}{4} + \frac{5}{4} = \frac{4.06+2.5+5}{4} = 2,89; \quad u_2^{(2)} = 5.41, \quad u_3^{(2)} = 2,91$$

$u_4^{(2)} = 5,39$ 3ème itération: $u_1^{(3)} = 3,33$, $u_2^{(3)} = 6,45$, $u_3^{(3)} = 3,94$, $u_4^{(3)} = 5,83$

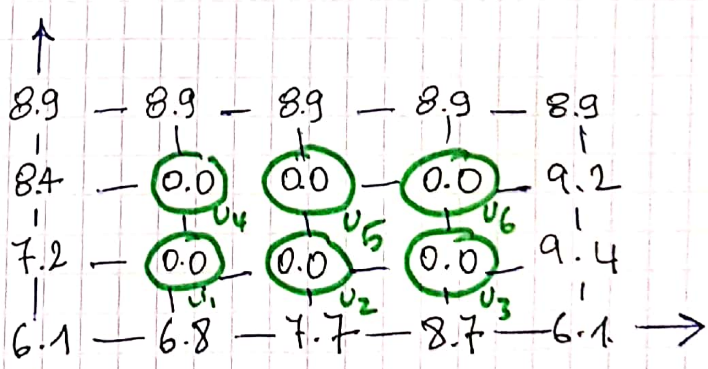
## Accelerating: Successive over relaxation (SOR)

$$u_{i,j}^{(k+1)} = \frac{u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k+1)}}{4} + u_{i,j}^{(k)} - u_{i,j}$$

$$\Rightarrow u_{i,j}^{(k+1)} = u_{i,j}^{(k)} + \underbrace{\left[\frac{u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k+1)} - 4u_{i,j}}{4}\right]}_{\text{Residuel}} w$$

w relaxation parameter $0.1 < w < 0.2$

Interpolation:



$u_1$: $u_2 + 7.2 + 6.8 + u_4 - 4u_1 = 0$

$-4u_1 + u_2 + u_4 = -14$

$u_2$: $u_1 + u_3 + u_5 - 4u_2 = -7.7$

$u_3$: $u_2 + 9.4 + 8.7 + u_6 - 4u_3 = 0$

$u_2 - 4u_3 + u_6 = -18.1$

$u_4$: $u_1 + 8.9 + u_5 + 8.4 - 4u_4 = 0$

$u_1 - 4u_4 + u_5 = -17.3$

$u_5$: $u_6 + u_4 + u_2 + 8.9 - 4u_5 = 0$

$u_2 + u_4 - 4u_5 + u_6 = -8.9$

$u_6$: $9.2 + u_5 + u_3 + 8.9 - 4u_6 = 0$

$u_3 + u_5 - 4u_6 = -18.1$

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} -14 \\ -7.7 \\ -18.1 \\ -17.3 \\ -8.9 \\ -18.1 \end{bmatrix}$$

Jacobi méthode: est méthode indirect pour résoudre un système linéaire. pour 2 successive iteration

- Jacobi method is inefficient because it does not make use of nodes that have already been updated. In which

Gauss-Seidel Méthod: uses that values of nodes that have been updated during the current step.

- Using row-by-column order, nodes to the west and south ($u_{i-1,j}$ and $u_{i,j-1}$) have been updated,

$$u_{i,j}^{k+1} = \frac{u_{i,j-1}^{k+1} + u_{i-1,j}^{k+1} + u_{i+1,j}^{k} + u_{i,j+1}^{k}}{4}$$

using row-by-column Order.

$$u_1^1 = \frac{6.8 + 7.2 + 0 + 0}{4} = 3.5$$

$$u_2^1 = \frac{7.7 + 3.5 + 0 + 0}{4} = 2.8$$

$$u_3^1 = \frac{2.8 + 9.4 + 8.7 + 0}{4} = 5{,}225$$

$$u_4^1 = \frac{8.4 + 8.9 + 3.5 + 0}{4} = 5{,}2$$

$$u_5^1 = \frac{5.2 + 2.8 + 8.9 + 0}{4} = 4{,}225$$

$$u_6^1 = \frac{9.2 + 4{,}225 + 5.225 + 8{,}9}{4} = 6{,}8875$$

$$u_1^2 = \frac{6.8 + 5.2 + 2.8 + 7.2}{4} = 5.5$$

⋮

$u_1^{17} = u_1^{18} = 7.639089$

$u_2^{17} = u_2^{18} = 8.176397$

$u_3^{16} = u_3^{17}$; $u_3^{18} = 8{,}785756$; $u_4^{17} = u_4^{18} = 8{,}379958$

$u_5^{17} = u_4^{18} = 8{,}580745$

$u_6^{16} = u_5^{17} = u_5^{18} = 8.866625$

$u_6$

| | | | | |
|---|---|---|---|---|
| 8.9 | 8.9 | 8.9 | 8.9 | 8.9 |
| 8.4 | $u_4$ =0 | $u_5$ | $u_6$ | 9.2 |
| 7.2 | $u_1$ | $u_2$ =0 | $u_3$ | 9.4 |
| 6.1 | 6.8 | 7.7 | 8.7 | 6.1 |

| | | | | |
|---|---|---|---|---|
| 8.9 | 8.9 | 8.9 | 8.9 | 8.9 |
| 8.4 | $u_4$ | $u_5$ =0 | $u_6$ =0 | 9.2 |
| 7.2 | 3.5 | $u_2$ | $u_3$ | 9.4 |
| 6.1 | 6.8 | 7.7 | 8.7 | 6.1 |

| | | | | |
|---|---|---|---|---|
| 8.9 | 8.9 | 8.9 | 8.9 | 8.9 |
| 8.4 | $u_4$ | $u_5$ | $u_6$ =0 | 9.2 |
| 7.2 | 3.5 | 2.8 | $u_3$ | 9.4 |
| 6.1 | 6.8 | 7.7 | 8.7 | 6.1 |

| | | | | |
|---|---|---|---|---|
| 8.9 | 8.9 | 8.9 | 8.9 | 8.9 |
| 8.4 | $u_4$ | $u_5$ | $u_6$ | 9.2 |
| 7.2 | 3.5 | 2.8 | 5.225 | 9.4 |
| 6.1 | 6.8 | 7.7 | 8.7 | 6.1 |

| | | | | |
|---|---|---|---|---|
| 8.9 | 8.9 | 8.9 | 8.9 | 8.9 |
| 8.4 | 5.2 | $u_5$ | $u_6$ | 9.2 |
| 7.2 | 3.5 | 2.8 | 5.225 | 9.4 |
| 6.1 | 6.8 | 7.7 | 8.7 | 6.1 |

| | | | | |
|---|---|---|---|---|
| 8.9 | 8.9 | 8.9 | 8.9 | 8.9 |
| 8.4 | 5.2 | 4.225 | $u_6$ | 9.2 |
| 7.2 | 3.5 | 2.8 | 5.225 | 9.4 |
| 6.1 | 6.8 | 7.7 | 8.7 | 6.1 |

**Résumé :** la dernière étape Consiste à resoudre le système d'équations en calculant $\{u\}$.

Les méthodes de résolution sont classées en deux types :

**Les méthodes directes :** Comme la méthode d'élimination substitution de Gauss

**Les méthodes indirectes (itérative)** comme la méthode de Gauss-Seidel ou encore les méthode de relaxation ou sur-relaxation

*** **Gauss-Seidel :** le principe de la méthode est l'utilisation de la solution la plus récente au cours de chaque itération

Gauss Carl Friedrich ; 1777 - 1855, mathématicien Allemand.

$$a_{11}T_1 + a_{12}T_2 + a_{13}T_3 = b_1$$
$$a_{21}T_1 + a_{22}T_2 + a_{23}T_3 = b_2$$
$$a_{31}T_1 + a_{32}T_2 + a_{33}T_3 = b_3$$

**a) Initiali~~ment~~sation :**
On se donne au départ une solution quelconque (~~initiale~~)
$T_1^{(0)}$, $T_2^{(0)}$, $T_3^{(0)}$ une solut° Connue à l'étape initiale $k=0$

**b) 1ère Itération :** les équations sont résolues successivement pour les valeurs inconnues de diagonale selon la procédure

$$T_1^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}T_2^{(0)} - a_{13}T_3^{(0)}\right)$$
$$T_2^{(1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}T_1^{(1)} - a_{23}T_3^{(0)}\right)$$
$$T_3^{(1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}T_1^{(1)} - a_{32}T_2^{(1)}\right)$$

**c) 2ème itération :**

$$T_1^{(2)} = \frac{1}{a_{11}}\left(b_1 - a_{12}T_2^{(1)} - a_{13}T_3^{(1)}\right)$$
$$T_2^{(2)} = \frac{1}{a_{22}}\left(b_2 - a_{21}T_1^{(2)} - a_{23}T_3^{(1)}\right)$$
$$T_3^{(2)} = \frac{1}{a_{33}}\left(b_3 - a_{31}T_1^{(2)} - a_{32}T_2^{(2)}\right)$$

**d)** Un calcul similaire est obtenu à l'itérat° $k+1$ on a :

$$T_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}T_2^{(k)} - a_{13}T_3^{(k)}\right)$$
$$T_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}T_1^{(k+1)} - a_{23}T^{(k)}\right)$$
$$T_3^{(k+1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}T_1^{(k+1)} - a_{32}T_2^{(k+1)}\right)$$

**e)**

**e)** La $i$ième équation donne :

$$T_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}T_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}T_j^{(k)}\right] \quad ***$$

On arrête le calcul lorsque deux valeurs successives de $T_i$ sont suffisament voisines. On peut utiliser les deux critères suivants:

**Convergence Absolue:** $\left\| T_i^{(k)} - T_i^{(k-1)} \right\| \leq \varepsilon$

**Convergence relative:** $\left\| \dfrac{T_i^{(k)} - T_i^{(k-1)}}{T_i^{(k)}} \right\| \leq \varepsilon$

La convergence ne dépend pas de la solution initiale mais des valeurs des coefficients $a_{ij}$.

La convergence est assurée si pour chaque ligne si :

$$a_{ii} \geq \sum_{j=1,\, i\neq j}^{n} |a_{ij}|$$

## Méthodes de Relaxation.

La méthode de Gauss-Seidel ne converge pas très rapidement, on utilise des méthodes de relaxation comme la méthode **SOR** "Successive Over Relaxation" ou la méthode Sur-Relaxation. {l'ingénieur Richard Southwell} utilisée pour accélerer la convergence.

La méthode prend son nom "relaxation", à partir de la façon dont on change la solution $x_i$ pour rendre le résidu $R_i$ égale à 0 (zéro). On dit relaxer le résidu.

$$R_i = a_{i1} T_1^{(k)} + a_{i2} T_2^{(k)} + \dots a_{in} T_n^{(k)} - b_i \neq 0$$

Partant de la solution initiale $T_i^{(0)}$, on obtient $T_i^{(1)}$ au lieu de réutiliser $T_i^{(1)}$ pour l'itération suivante, on voit que la convergence serait plus rapide si on insérait :

$$\overline{T}_i^{(1)} = T_i^{(0)} + \omega \left( T_i^{(1)} - T_i^{(0)} \right) \qquad \omega : \text{facteur de relaxation}$$

$$\overline{T}_i^{(k+1)} = T_i^{(k)} + \omega \left( T_i^{(k+1)} - T_i^{(k)} \right) \Rightarrow \overline{T}_i^{(k+1)} = (1-\omega) T_i^{(k)} + \omega T_i^{(k+1)}$$

En substituant $T_i^{(k+1)}$ dans l'équation ***

$$\overline{T}_i^{(k+1)} = \omega \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij} T_j^{(k+1)} - \sum_{j=i+1}^{m} a_{ij} T_j^{(k)} \right) + (1-\omega) T_i^{(k)}$$

Selon la valeur du facteur de relaxation $w$, on a les méthodes :

$\omega = 1$ Méthode de Gauss-Seidel

$1 < w < 2$ Méthode de sur-Relaxation (SOR)

$w < 1$ Méthode de Sous-Relaxation

On a vu que l'équation aux différences finies de Laplace pour $r = 1$

$$T_{i-1,j} - 4T_{i,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1} = 0$$

En résolvant pour $T_{i,j}$

$$T_{i,j} = 0.25\left(T_{i-1,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1}\right)$$

utilisant l'équation : $\bar{T}_i^{(k+1)} = (1-w)\,T_i^{(k)} + w\,T_i^{(k+1)}$ on peut écrire la solution à l'itération $k+1$

$$\bar{T}_{i,j}^{(k+1)} = 0.25\left(T_{i-1,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1}\right)^{(k+1)} + (1-w)\,T_{i,j}^{(k)}$$

D'où l'algorithme de résolution de l'équation de Laplace par la méthode SOR.

$T(i,j) = T_{init}$ (aux point pivot)

Conditions aux frontières pour obtenir $b_i$

Pour $k = 1 ; k_{max}$

pour $j = 2 ; n-1$

pour $i = 2 ; m-1$

$$\bar{T}_{i,j}^{(k+1)} = 0{,}25\,w\left(T_{i-1,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1}\right)^{(k+1)} + (1-w)\,T_{i,j}^{(k)}$$

Fin

Fin

Fin

# Série de Maclaurin & Taylor.

① la Série de Maclaurin de la fonction $\sin(2x)$.

$f(x) = \sin(2x)$
$f'(x) = 2\cos(2x)$
$f''(x) = -4\sin(2x)$
$f'''(x) = -8\cos(2x)$
$f^{(4)}(x) = 16\sin(2x)$
$f^{(5)}(x) = 32\cos(2x)$
$\sin(0) = 0$
$\cos(0) = 1$

$\sin(2x) = ?$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}}{5!}x^5 + \dots$$

$$\sin(2x) = \sin(0) + 2\cos(0)\cdot\frac{x}{1!} + (-4)\sin(0)\frac{x^2}{2!} + (-8)\cos(0)\frac{x^3}{3!} + 16\sin(0)\frac{x^4}{4!} + 32\cos(0)\frac{x^5}{5!}$$

$$\sin(2x) = 2x - 8\frac{x^3}{3!} + 32\frac{x^5}{5!} = 2x - 8\frac{x^3}{6} + \frac{32}{120}x^5 + \dots$$

$$\sin(2x) = 2x - 1{,}333x^3 + 0.26667x^5 + \dots$$

le coefficient de terme $x^5$ est : $0.26667$ #

② $f(3) = 6$ ; $f'(3) = 8$ ; $f''(3) = 11$ ; $f^{(n\geq 3)}(3) = 0$    $f(x)$ continue et $\in [3,7]$

$x_0 = 3 > 0 \Rightarrow$ formule de Taylor.

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \frac{(x-x_0)^3}{3!}f'''(x_0) + \dots$$

$x = 3 \Rightarrow (x - x_0) = (7-3) = 4$ (pas).

$$f(3+4) = f(3) + 4f'(3) + \frac{4^2}{2!}f''(3) + \frac{4^3}{3!}f'''(3) + \dots$$

$$= f(3) + 4f'(3) + 8f''(3) + 0$$

$$= 6 + 4(8) + 8(11) = 126 \ \#$$

③ $\frac{dy}{dx} = y^3 + 2$ et $y(0) = 3$

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0)$$

$y(0,2) = ?$   on a $x_0 = 0 \Rightarrow x - x_0 = 0,2 - 0 = 0,2$

$$y(0,2) = y(0) + (0,2)y'(0) + \frac{(0,2)^2}{2!}y''(0)$$

donnée $\Rightarrow y(0) = 3$

$y'(x) = \frac{dy}{dx} = y^3 + 2 \Rightarrow y'(0) = [y(0)]^3 + 2 = 3^3 + 2 = 29$ #

$y''(x) = \frac{d^2y}{dx^2} = 3y^2\frac{dy}{dx}$ ~~$\Rightarrow y''(0) = 3\cdot[y(0)]^2 = 3\times($~~

$y''(x) = 3y^2\cdot\frac{dy}{dx}$
$= 3y^2[y^3+2] \Rightarrow y''(0) = 3[y(0)]^2[(y(0))^3 + 2]$
$= 3(3^2)(3^3+2) = 3\cdot 9\cdot 29 = 783$ #

$$y(0.2) = 3 + 0.2(29) + 0.02(783) = 24.46 \ \#$$

Scanned with CamScanner

# Méthode des différences finies.

① EDP : $A\dfrac{\partial^2 u}{\partial x^2} + B\dfrac{\partial^2 u}{\partial x \partial y} + C\dfrac{\partial^2 u}{\partial y^2} + D = 0$

est Elliptique si $B^2 - 4(A)(C) < 0$.

② $\dfrac{d^2y}{dx^2} = 6x - 0{,}5x^2$ ; $y(0) = 0$ ; $y(12) = 0$ et $\Delta x = h = 4$.

$\xleftrightarrow{\Delta x}\xleftrightarrow{\Delta x}$

$i-1 \quad i \quad i+1$

$\dfrac{d^2y}{dx^2}$ pour le noeud $i$, par la méthode des différences finies.

pour $x_i = x_1$ :

$\dfrac{d^2y}{dx^2} \simeq \dfrac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$ , on cherche $y(4)$.

$i-1 \quad i \quad i+1$

$y(0) \quad y(4) \quad y(8)$

$\begin{cases} y_{i+1} = y(8) \\ y_i = y(4) \\ y_{i-1} = y(0) \end{cases}$

$\boxed{\dfrac{d^2y}{dx^2} = \dfrac{y(8) - 2y(4) + y(0)}{16}}$ #

$\dfrac{y(8) - 2y(4) + y(0)}{16} = 6x_i - 0{,}5x_i^2$

$\dfrac{y(8) - 2y(4) + y(0)}{16} = 6x_i - 0{,}5x_i^2$

$y(8) - 2y(4) + y(0) = 6x_i \cdot 16 - 0{,}5 \cdot 16\, x_1^2 = 96x_1 - 8x_1^2$

$0 \quad 4 \quad 8 \quad 12$

$\xleftrightarrow{4}$

$x_0^0 \quad x_1^1 \quad x_2^2 \quad x_3^3 \quad x_4^4$

$y(8) - 2y(4) + y(0) = 96(4) - 8(4)^2$

$\boxed{y(8) - 2y(4) + y(0) = 256}$

pour $x_i = x_2$ : $i = 2$, $x_2 = 8$

$\dfrac{d^2y}{dx} = \dfrac{y(12) - 2y(8) + y(4)}{\Delta x^2} = y(12) - 2y(8) + y(4) / 16$.

$y(12) - 2y(8) + y(4) = 16 \cdot 6x_2 - 0{,}5 \cdot 16x_2^2 = 96x_2 - 8x_2^2$.

$\boxed{y(12) - 2y(8) + y(4) = 96(8) - 8(64) = 256}$

$\begin{cases} \cancel{y(0)} - 2y(4) + y(8) = 256 \\ y(4) - 2y(8) + \cancel{y(12)} = 256 \end{cases}$ $\qquad \begin{cases} -2y(4) + y(8) = 256 \\ y(4) - 2y(8) = 256 \end{cases}$

$\begin{bmatrix} -2 & +1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y(4) \\ y(8) \end{bmatrix} = \begin{bmatrix} 256 \\ 256 \end{bmatrix}$

$\boxed{\begin{aligned} y(4) &= -256 \\ y(8) &= -256 \end{aligned}}$

(3) $x^3 \frac{\partial^2 u}{\partial x^2} + 27 \frac{\partial^2 u}{\partial y^2} + 3 \frac{\partial^2 u}{\partial x \partial y} + 5u = 0$ dans quelle region Elliptique

EDP : $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$ Elliptiq : $B^2 - 4(A)(C) < 0$

$A = x^3$ ; $B = 3$ ; $C = 27$

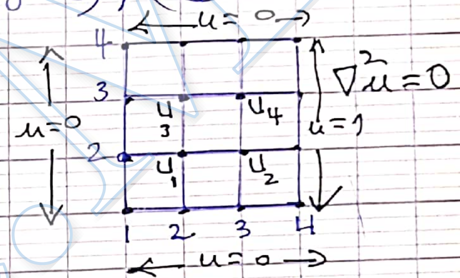
$\Delta = B^2 - 4(A)(C) = (3)^2 - 4(x^3)(27) = 9 - 108x^3 < 0$

$-108x^3 < -9 \iff -x^3 < -\frac{9}{108} \Rightarrow x^3 > \frac{1}{12} \Rightarrow x > \left[\frac{1}{12}\right]^{1/3}$

(4) $\frac{\partial^2 u}{\partial x^2} = u(x+\Delta x, y) - 2u(x,y) + u(x-\Delta x, y) / (\Delta x)^2$

$\frac{\partial^2 u}{\partial y^2} = u(x, y+\Delta y) - 2u(x,y) + u(x, y-\Delta y) / (\Delta y)^2$

(5) $u_{i,j}$ par différences finis pour $h = k = 1/3$.




$u_{i,j} = \frac{1}{4}\left[u_{i+1,j} - u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right]$

$u_1 = [u_2 + 0 + u_3 + 0]/4$ $\quad u_3 = [u_4 + 0 + 0 + u_1]/4$

$u_2 = [u_1 + 1 + u_4 + 0]/4$ $\quad u_4 = [u_2 + 1 + 0 + u_3]/4$

$u_1 = (u_2 + u_3)/4$ ; $u_2 = (u_1 + u_4 + 1)4$ ; $u_3 =$

$u_3 = (u_4 + u_1)/4$ ; $u_4 = (u_2 + u_3 + 1)/4$.

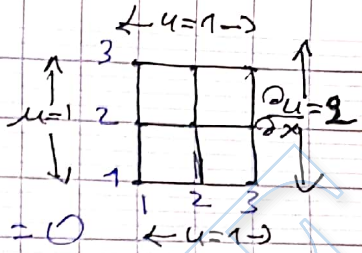
$4u_1 - u_2 - u_3 = 0$ ; $4u_2 - u_1 - u_4 - 1 = 0$

$4u_3 - u_4 - u_1 = 0$ ; $4u_4 - u_2 - u_3 - 1 = 0$

$$\begin{cases} 4u_1 - u_2 - u_3 = 0 \\ -u_1 + 4u_2 - u_4 = 1 \\ -u_1 + 4u_3 - u_4 = 0 \\ -u_2 - u_3 + 4u_4 = 1 \end{cases} \qquad \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$u_1 = 1/8$ ; $u_2 = 3/8$ , $u_3 = 1/8$ et $u_4 = 3/8$

⑥ Résoudre $\nabla^2 u = 0$

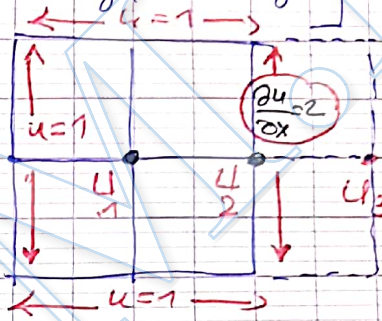



$$\nabla^2 u = 0 \Leftrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0$$

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{\Delta y^2} = 0$$

$$\Delta x = \Delta y \Rightarrow u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right]$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{u_{i+1} - u_{i-1}}{2 \times \Delta x}} \quad \boxed{\frac{\partial u}{\partial x} = \frac{u_3 - u_1}{2 \times 0{,}5} = 2}$$




$$u_1 = [u_2 + 1 + 1 + 1]/4 \Rightarrow 4u_1 - u_2 = 3$$

$$u_2 = [u_3 + u_1 + 1 + 1]/4 \Rightarrow 4u_2 - u_3 - u_1 = 2$$

$$\frac{\partial u}{\partial x} = \frac{u_3 - u_1}{1} = 2 \Rightarrow \boxed{u_3 = 2 + u_1}$$

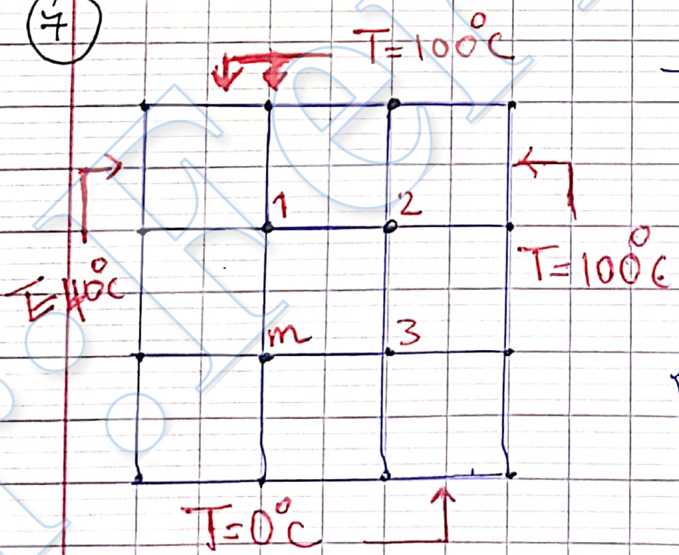
$$4u_1 - u_2 = 3$$

$$4u_2 - (u_1 + 2) - u_1 = 2 \Leftrightarrow 4u_2 - 2u_1 = 4$$

$$\begin{cases} 4u_1 - u_2 = 3 \\ -1u_1 + 2u_2 = 2 \end{cases}$$

$$\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad u_1 = \frac{8}{7} \qquad u_2 = \frac{11}{7}$$

⑦




$T_m = 50°C$ ; $\Delta x = \Delta y$, $\nabla^2 u = 0$

$$\begin{cases} 4T_1 = T_2 + 100 + 40 + T_m \\ 4T_2 = 100 + 100 + T_1 + T_3 \\ 4T_3 = 100 + T_2 + 0 + T_m \end{cases}$$

$$\begin{cases} 4T_1 - T_2 = 190 \\ -T_1 + 4T_2 - T_3 = 200 \\ -T_2 + 4T_3 = 150 \end{cases}$$

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 190 \\ 200 \\ 150 \end{bmatrix}$$

$T_1 = 475/7$

$T_2 = 570/7$

$T_3 = 405/7$

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