حل السلسلة (2)

(التكامل و الدوال الأصلية)

التمرين ①:

$\int x^2 e^x dx = x^2 e^x - 2\int x e^x dx$ $u = x^2 \to u' = 2x$, $v' = e^x \to v = e^x$

$= x^2 e^x - 2(x e^x - e^x) + c$

$= e^x(x^2 - 2x + 2) + c, \quad c \in \mathbb{R}$

$\int \ln(x + \sqrt{x^2+1})\,dx = x \ln(x + \sqrt{x^2+1}) - \int \frac{x}{\sqrt{x^2+1}} dx$ $u = \ln(x+\sqrt{x^2+1}) \to u' = \frac{1}{\sqrt{x^2+1}}$, $v' = 1 \to v = x$

$= x \ln(x + \sqrt{x^2+1}) - \sqrt{x^2+1} + c, \quad c \in \mathbb{R}$

$\int \frac{x^2}{(x^2+1)^2} dx = -\frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\int \frac{1}{x^2+1} dx$ $u = x \to u' = 1$, $v' = \frac{x}{(x^2+1)^2} \to v = -\frac{1}{2}\frac{1}{x^2+1}$

$= -\frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\operatorname{arctg} x + c, \quad c \in \mathbb{R}$

$\int_0^{\pi} (x^2+x+1)\sin x\,dx = -\left[(x^2+x+1)\cos x\right]_0^{\pi} + \int_0^{\pi}(2x+1)\cos x\,dx$, $u = x^2+x+1 \to u' = 2x+1$, $v' = \sin x \to v = -\cos x$

$= -\left[(x^2+x+1)\cos x\right]_0^{\pi} + \left[(2x+1)\sin x\right]_0^{\pi} - 2\int_0^{\pi}\sin x\,dx$

$= -\left[(x^2+x+1)\cos x\right]_0^{\pi} + \left[(2x+1)\sin x\right]_0^{\pi} + 2\left[\cos x\right]_0^{\pi} = \pi^2 + \pi - 2$ …

التمرين ②:

1) نضع $t = e^x$ لدينا: $\int_0^1 \frac{dx}{1+e^x} = \int_1^e \frac{dt}{t(t+1)} = \int_1^e \left(\frac{1}{t} - \frac{1}{t+1}\right)dt = 1 - \ln(e+1) + \ln 2$

2) لدينا: $x^2+6x+10 = (x+3)^2+1$ نضع $t = x+3$ لدينا: $\int \frac{dx}{x^2+6x+10} = \int \frac{dt}{t^2+1} = \operatorname{arctg} t + c = \operatorname{arctg}(x+3) + c, \quad c \in \mathbb{R}$

3) نضع $t = \operatorname{tg}\frac{x}{2}$ لدينا $\cos x = \frac{1-t^2}{1+t^2}$ و $dx = \frac{2dt}{1+t^2}$ ومنه:

$\int \frac{dx}{2+\cos x} = 2\int \frac{dt}{3+t^2} = \frac{2\sqrt{3}}{3}\operatorname{arctg}\left(\frac{\sqrt{3}}{3}\operatorname{tg}\frac{x}{2}\right) + c, \quad c \in \mathbb{R}$

4) نضع $t = \operatorname{th}\frac{x}{2}$ لدينا $\operatorname{sh} x = \frac{2t}{1-t^2}$ ، $\operatorname{ch} x = \frac{1+t^2}{1-t^2}$ و $dx = \frac{2dt}{1-t^2}$

$\int \frac{dx}{\operatorname{ch}x\,\operatorname{sh}x} = \int \frac{1-t^2}{t(t^2+1)}dt = \int\left(\frac{1}{t} - \frac{2t}{t^2+1}\right)dt = \ln\left|\operatorname{th}\frac{x}{2}\right| - \ln\left(1+\operatorname{th}^2\frac{x}{2}\right) + c$ …

التمرين ③:

$\int \frac{3x^3-2x^2+2x+2}{x^2-x+1}dx = \int \frac{(x^2-x+1)(3x+1)+1}{x^2-x+1}dx$

$= \int (3x+1)\,dx + \int \frac{dx}{x^2-x+1} = \int (3x+1)\,dx + \int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}$

$= \frac{3}{2}x^2 + x + \frac{2\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2\sqrt{3}}{3}x - \frac{\sqrt{3}}{3}\right) + c, \quad c \in \mathbb{R}$

2) $\int \frac{2x+1}{x^3-3x^2-4x}\,dx = \int \frac{2x+1}{x(x+1)(x-4)}\,dx$

$= \int \left(\frac{-\frac{1}{4}}{x} + \frac{-\frac{1}{5}}{x+1} + \frac{\frac{9}{20}}{x-4} \right) dx = -\frac{1}{4}\ln|x| - \frac{1}{5}\ln|x+1| + \frac{9}{20}\ln|x-4| + c,\ c\in\mathbb{R}$

3) $\int \frac{dx}{x^3+2x^2+x} = \int \left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} \right) dx = \ln|x| - \ln|x+1| + \frac{1}{x+1} + c,\ c\in\mathbb{R}$

4) $\int \frac{dx}{x^3+1} = \int \frac{dx}{(x+1)(x^2-x+1)}$

$= \frac{1}{3}\int \left(\frac{1}{x+1} - \frac{x-2}{x^2-x+1} \right) dx$

$= \frac{1}{3}\int \left(\frac{1}{x+1} - \frac{1}{2}\frac{2x-1}{x^2-x+1} + \frac{3}{2}\frac{1}{x^2-x+1} \right) dx$

$= \frac{1}{3}\int \left(\frac{1}{x+1} - \frac{1}{2}\frac{2x-1}{x^2-x+1} + \frac{3}{2}\frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}} \right) dx$

$= \frac{1}{3}\left[\ln|x+1| - \frac{1}{2}\ln(x^2-x+1) + \frac{3}{2}\cdot\frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2}{\sqrt{3}}(x-\frac{1}{2}) \right] + c,\ c\in\mathbb{R}$

$= \frac{1}{3}\left[\ln|x+1| - \frac{1}{2}\ln(x^2-x+1) + \sqrt{3} \operatorname{arctg} \frac{2\sqrt{3}}{3}(x-\frac{1}{2}) \right] + c,\ c\in\mathbb{R}$

ت (4):

1) نضع $t = e^x$ عندئذ $\int \frac{e^{3x}}{1+e^{2x}}\,dx = \int \frac{t^2}{1+t^2}\,dt = e^x - \operatorname{arctg} e^x + c,\ c\in\mathbb{R}$

2) نضع $t = x^{\frac{1}{6}}$ عندئذ $dx = 6t^5\,dt$ ونجد:

$\int \frac{\sqrt{x}-\sqrt[3]{x}}{\sqrt{x}+\sqrt[3]{x}}\,dx = 6\int \frac{t^6-t^5}{t+1}\,dt = 6\int \left(t^5 - 2t^4 + 2t^3 - 2t^2 + 2t - 2 + \frac{2}{t+1} \right) dt$

$= 6\left[\frac{x}{6} - \frac{2}{5}\sqrt[6]{x^5} + \frac{1}{2}\sqrt[3]{x^2} - \frac{2}{3}\sqrt{x} + \sqrt[3]{x} - 2\sqrt[6]{x} + 2\ln(\sqrt[6]{x}+1) \right] + c,\ c\in\mathbb{R}$

ت (5):

1) $I_0 = \int_1^e x\,dx = \frac{e^2}{2} - \frac{1}{2}$

$I_1 = \int_1^e x \ln x\,dx = \frac{e^2}{4} + \frac{1}{4}$

2) لدينا: $I_{n+1} = \int_1^e x(\ln x)^{n+1}\,dx$

$u = (\ln x)^{n+1} \longrightarrow u' = (n+1)\frac{1}{x}(\ln x)^n$

$v' = x \longrightarrow v = \frac{x^2}{2}$

$I_{n+1} = \frac{e^2}{2} - \frac{n+1}{2} I_n$

وبالتالي ينتج:

$I_2 = \frac{e^2}{2} - I_1 = \frac{e^2}{4} - \frac{1}{4}$

$I_3 = \frac{e^2}{2} - \frac{3}{2} I_2 = \frac{e^2}{8} + \frac{3}{8}$