Ministry of high education and scientific researches

University Of El Oued

Introduction on linear operator

and

Spectral Theory

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Contents

University of El Oued
Exact Sciences Faculty
Department of Mathematics

Introduction to linear bounded operators and spectral theory

A course for Master I^{st}

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Chapter 1

Bilinear and quadratic Forms

1.1 Bilinear and sesquilinear forms

Throughout this paragraph, we denote by V a vector (linear) space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).

Definition 1.1. *A bilinear form over V is a two-variables functional $B : V \times V \longrightarrow \mathbb{K}$, such that for all $\alpha, \beta \in \mathbb{K}$ and all $x, y, z \in V$, we have*

i) $B(\alpha x + y, z) = \alpha B(x, z) + B(y, z)$,

ii) $B(x, \beta y + z) = \beta B(x, y) + B(x, z)$..

In other words, $B(\cdot, \cdot)$ is linear with respect to each of its components.
If B satisfies the above statement i) and

iii) $B(z, \alpha x + y) = \overline{\alpha} B(z, x) + B(z, y)$, $\forall \alpha \in \mathbb{C}, \forall x, y, z \in V$,

then B is called a sesquilinear form.

Example 1.1. *The simplest example is the functions*

$$\begin{aligned} B : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} & \qquad S : \mathbb{C} &\longrightarrow C \\ : (x, y) &\longrightarrow xy, & : (x, y) &\longrightarrow x\overline{y}. \end{aligned}$$

B is bilinear form and S is sesquilinear form.

Example 1.2. *Suppose that V is a vector space and let B be the inner product over V, $B(x, y) = \langle x, y \rangle$, then, B is a bilinear form if $\langle \cdot, \cdot \rangle$ is a real valued inner product, and a sesquilinear form if $\langle \cdot, \cdot \rangle$ is complex valued inner product.*

Example 1.3. *Let V be an $n-$dimensional vector space, and let A be a square matrix over $\mathbb{K}$, $A \in \mathcal{M}_n(\mathbb{K})$. For every $u, v \in V$, we set*

$$B(u, v) = u^T A v = \sum_{1 \leq i,j \leq n} u_i a_{ij} v_j,$$

where, u^T is the transpose of u. Then, B is a bilinear form over V.

Similarly, $S(u,v) = u^T A\overline{v} = \sum_{1\le i,j\le n} u_i a_{ij}\overline{v_j}$, is a sesquilinear form.

Lemma 1.1. *Let B be a bilnear or a sesquilinear form, then $B(0,y) = B(x,0)$, for all $x, y \in V$.*

Proof. $B(0,y) = B(0+0,y) = 2B(0,y)$, then $B(0,y) = 0$. □

Proposition 1.1. *Let B be a bilinear form over a finite dimensional linear space V, then, there exists a matrix $A \in \mathcal{M}_n(\mathbb{R})$, such that*

$$B(u,v) = u^T Av.$$

Similarly, for every sesquilinear form S there exists a matrix $A \in \mathcal{M}_n(\mathbb{C})$, such that

$$S(u,v) = u^T A\overline{v}.$$

Demonstration. *Let (e_i) be a basis in V, then, it suffices to take $a_{ij} = B(e_i, e_j)$ in the first case and $a_{ij} = S(e_i, e_j)$ in the second case, and $A = (a_{ij})$.*

Definition 1.2. *A bilinear form B is said to be:*

1) symmetric if

$$B(x,y) = B(y,x), \forall x, y \in V,$$

2) skew-symmetric or anti-symmetric if

$$B(x,y) = -B(y,x), \forall x, y \in V.$$

Definition 1.3. *A sesquilinear form S is said to be:*

1) symmetric if

$$S(x,y) = \overline{S(y,x)}, \forall x, y \in V,$$

2) skew-symmetric or anti-symmetric if

$$S(x,y) = -\overline{S(y,x)}, \forall x, y \in V.$$

Definition 1.4. *A bilinear or a sesquilinear form B, is said to be*

1) alternating if

$$B(x,x) = 0, \ \forall x \in V,$$

2) non degenerate if

$$\forall x \in V - \{0\}, \exists y \in V : B(x,y) \neq 0,$$

then, a degenerate form is such that

$$\exists x \in V - \{0\}, \forall y \in V : B(x,y) = 0.$$

Example 1.4. *The real inner product over V is a symmetric bilinear non degenerate form.*

The bilinear form $B(u,v) = u^T Av$ is symmetric if and only if the matrix A is symmetric, and it is skew-symmetric if and only if A is skew-symmetric.

Definition 1.5. *A bilinear or a sesquilinear form B is said to be positive if*

$$B(x,x) \geq 0, \forall x \in V,$$

and it said to be definite if

$$B(x,x) > 0, \forall x \in V, x \neq 0.$$

Example 1.5. *Let $V = L^2(0,\pi)$ and*

$$B(f,g) = \int_0^\pi f(x)\overline{g(x)}dx,$$

we have,

$$B(f,f) = \int_0^\pi |f(x)|^2 dx > 0, \text{ for all } f \neq 0,$$

then, B is positive definite.

In what follows in this paragraph, we suppose that V is a normed space with norm $\|\cdot\|$.

Theorem 1.1. [***Uniform boundedness princible***] *Let X be a Banach space, Y a normed linear space, and let $T_\alpha : X \longrightarrow Y$, $\alpha \in I$, be a family of bounded linear operators, $T_\alpha \in \mathscr{L}(X,Y)$. Assume that the family $\{T_\alpha; \alpha \in I\}$ is pointwise bounded, that is,*

$$\forall x \in X, \exists C_x > 0 : \|T_\alpha x\| \leq C_x, \forall \alpha \in I.$$

Then $\{T_\alpha; \alpha \in I\}$ is uniformly bounded, that is,

$$\exists C > 0 : \|T_\alpha\| \leq C, \forall \alpha \in I.$$

Definition 1.6. *Let $B : V \times V \longrightarrow \mathbb{K}$, we say that B is continuous over V, if for all $(x,y) \in V \times V$ and for all $\varepsilon > 0$, there exists $\alpha > 0$,*

$$\forall (x',y') \in V \times V : \|(x,y) - (x',y')\| < \alpha \Longrightarrow |B(x,y) - B(x',y')| < \varepsilon.$$

Proposition 1.2. *Let $B : V \times V \longrightarrow \mathbb{K}$ then, the following statements are equivalent :*

i) B is continuous,
ii) B is continuous at $(0,0)$,
iii) $\exists C > 0$, such that

$$|B(x,y)| \leq C\|x\|\|y\|, \forall x, y \in V.$$

Demonstration. *$(i) \Longrightarrow (ii)$ evident.*

$(ii) \Longrightarrow (iii)$, suppose that B continuous in $(0,0)$, then from the definition we have

$$\forall \varepsilon > 0, \exists \alpha > 0 : \|(x,y)\|_{V \times V} < \alpha \Longrightarrow |B(x,y)| < \varepsilon.$$

If B is not continuous at $(0,0)$, then there exists $\varepsilon > 0$, such that $\forall\alpha > 0$, $\exists (x,y) \in V \times V$, $\|(x,y)\|_{V\times V} < \alpha$ and $|B(x,y)| > \varepsilon$.

Suppose that iii) is not satisfied, then,

$$\forall n \in \mathbb{N}^*, \exists x_n, y_n \in V : |B(x_n, y_n)| > n^2 \|x_n\| \|y_n\| .$$

Clearly, $x_n \neq 0$ and $y_n \neq 0$. Then, if we set $x_n^* = \frac{x_n}{n\|x_n\|}$ and $y_n^* = \frac{y_n}{n\|y_n\|}$, we have

$$|B(x_n^*, y_n^*)| = \frac{1}{n^2 \|x_n\| \|y_n\|} |B(x_n, y_n)| > 1.$$

Therefore, there exists $0 < \varepsilon < 1$, such that $\forall\alpha > 0$, there exist x_n^* and y_n^* such that

$$\|(x_n^*, y_n^*)\| = \sup(\|x_n^*\|, \|y_n^*\|) = \frac{1}{n} < \alpha, \text{ but } |B(x_n^*, y_n^*)| > 1 > \varepsilon,$$

which contradict (ii).

$(iii) \Longrightarrow (i)$. Suppose that (iii) is not satisfied and let (x_n) and (y_n) be two sequences from V such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$.

Thus, (x_n) and (y_n) are bounded, that there exists $M > 0$, such that $\|x_n\| < M$ and $\|y_n\| < M$. On the other hand, we have

$$\begin{aligned} |B(x_n, y_n) - B(x,y)| &\leq |B(x_n, y_n) - B(x_n, y)| + |B(x_n, y) - B(x,y)| \\ &\leq |B(x_n, y_n - y)| + |B(x_n - x, y)| \\ &\leq C\|x_n\| \|y_n - y\| + C\|y\| \|x_n - x\| \\ &\leq CM\|y_n - y\| + C\|y\| \|x_n - x\| \longrightarrow 0, \end{aligned}$$

which shows that B is continuous in each $(x,y) \in V \times V$, this completes the proof.

Lemma 1.2. *Suppose that V is a Banach space. Then, B is continuous if and only it is separately continuous, that is continuous in each coordinate.*

Demonstration. *It suffices to prove the sufficiency of the condition.*

Suppose that V is a Banach space and B is separately continuous. For every $y \in V$ with $\|y\| = 1$, let $T_y = B(\cdot, y) : V \longrightarrow \mathbb{K}$. Then, $\{T_y; \|y\| = 1\}$ is a family of bounded operator. Moreover, for any fixed $x \in V$, $\{T_y x = B(x,y); \|y\| = 1\}$ is bounded, because

$$\|T_y x\| = \|B(x,y)\| \leq C_x \|y\| = C_x.$$

This means that the family $\{T_y; \|y\| = 1\}$ is pointwise bounded. Thus, from the uniform boundedness principle, there exixts $C > 0$, such that

$$\|T_y\| \leq C, \forall y \in V, \|y\| = 1.$$

Consequently, for every $x \in V$,

$$\|B(x,y)\| = \|T_y x\| \leq C\|x\| .$$

Therefore, for every, $z \in V, z \neq 0,$

$$\begin{aligned}\|B(x,z)\| &= \left\| \|z\| B\left(x, \frac{z}{\|z\|}\right)\right\| = \left\|B\left(x, \frac{z}{\|z\|}\right)\right\| \|z\| \\ &\leq C\|x\|\|z\|,\end{aligned}$$

which proves that B *is continuous.*

Example 1.6. *Let* B *be the bilinear form on* $H^1(0,\pi)$ *defined by*

$$B(u,v) = \int_0^\pi u_x v_x dx + \int_0^\pi uv dx.$$

$H^1(0,\pi)$ *is endowed with the norm*

$$\|u\|_2 = \|u\|_1 + \|u_x\|_1.$$

We have,

$$\begin{aligned}|B(u,v)| &\leq \int_0^\pi |u_x v_x| dx + \int_0^\pi |uv| dx \\ &\leq \left(\int_0^\pi u_x^2 dx\right)^{\frac{1}{2}} \left(\int_0^\pi v_x^2 dx\right)^{\frac{1}{2}} + \left(\int_0^\pi u^2 dx\right)^{\frac{1}{2}} \left(\int_0^\pi v^2 dx\right)^{\frac{1}{2}} \\ &\leq \|u_x\|\|v_x\| + \|u\|\|v\| \leq (\|u_x\| + \|u\|)(\|v_x\| + \|v\|).\end{aligned}$$

Consequently, B *is continuous.*

Remark 1.1. *The reason for the name bounded for a continuous bilinear form* B*, is justified from the fact that* B *transform a bounded set* $S = \{x \in V;\ \|x\| \leq M\}$ *of* V *into a bounded set* $\{r \in \mathbb{R}; |r| \leq CM^2\}$ *in* $\mathbb{R}$.

Definition 1.7. *A bilinear form* B *on a normed vector space* $(V, \|\cdot\|)$ *is said to be elliptic, or coercive, if there is a positive constant* $\alpha > 0$*, such that*

$$B(x,x) \geq \alpha \|x\|^2,\ \forall x \in V.$$

Example 1.7. *Let* I *be an interval in* $\mathbb{R}$*, and* p, q *be two functions that satisfy*

$$p \in C^1\left(\overline{I}\right),\ q \in C(I)$$

and there exists $\alpha > 0$*, such that* $p(x) \geq \alpha$*, for all* $x \in \overline{I}$.

Set $V = H_0^1(I)$ *endowed with the norm* $\|u\|_{H_0^1} = \sqrt{\int_I u_x^2(x)\,dx}$*, finally, let* B *be defined on* $V \times V$ *by*

$$B(u,v) = \int_I p(x)\,u_x(x)\,v_x(x)\,dx + \int_I q(x)\,u(x)\,v(x)\,dx.$$

If $q \geq 0$*, then* B *is coercive. Indeed*

$$\begin{aligned}B(u,u) &= \int_I p(x)\,u_x^2(x)\,dx + \int_I q(x)\,u^2(x)\,dx \\ &\geq \alpha \int_I u_x^2(x)\,dx = \alpha \|u\|_{H_0^1}^2.\end{aligned}$$

1.2 Quadratic forms

Definition 1.8. *A quadratic form over V is a function $q : V \longrightarrow \mathbb{K}$ that satisfies the two following statements :*

i)

$$q(\lambda x) = \lambda^2 q(x), \forall x \in V, \forall \lambda \in \mathbb{K}, \tag{1.1}$$

ii) the form $\widetilde{B} : V \times V \longrightarrow \mathbb{K}$ defined by

$$\widetilde{B}(x, y) = q(x + y) - q(x) - q(y) \tag{1.2}$$

is bilinear.

The bilinear form $\widetilde{B}$ is called the underlying bilinear form of q (or the associated bilinear form of q). Note that $\widetilde{B}$ is always symmetric.

Example 1.8. *1) $V = \mathbb{R}$, $q(x) = \alpha x^2$,*

Example 1.9. *2) $V = \mathbb{R}^2$, $q(x, y) = ax^2 + bxy$,*

are quadratic forms on V.

1)

$$q(\lambda x) = \lambda^2 \alpha x^2 = \lambda^2 q(x).$$

$$\widetilde{B}(x, y) = \alpha (x + y)^2 - \alpha x^2 - \alpha y^2 = 2\alpha xy.$$

2)

$$q(\lambda(x, y)) = a(\lambda x)^2 + b(\lambda x)(\lambda y) = \lambda^2 \left(ax^2 + bxy\right) = \lambda^2 q(x, y),$$

$$\begin{aligned}\widetilde{B}((x, y), (u, v)) &= q(x + u, y + v) - q(x, y) - q(u, v)\\ &= a(x + u)^2 + b(x + u)(y + v) - ax^2 - bxy - au^2 - buv\\ &= 2axu + bxv + byu\end{aligned}$$

$$\begin{aligned}\widetilde{B}((x + x', y + y'), (u, v)) &= 2a(x + x')u + b(x + x')v + b(y + y')u\\ &= 2axu + bxv + byu + 2ax'u + bx'v + by'u\\ &= \widetilde{B}((x, y), (u, v)) + \widetilde{B}((x', y'), (u, v))\end{aligned}$$

$$\begin{aligned}\widetilde{B}((x, y), (u + u', v + v')) &= 2ax(u + u') + bx(v + v') + by(u + u')\\ &= 2axu + bxv + byu + 2axu' + bxv' + byu'\\ &= \widetilde{B}((x, y), (u, v)) + \widetilde{B}((x, y), (u', v'))\end{aligned}$$

$$\begin{aligned}\widetilde{B}((\alpha x, \alpha y), (u, v)) &= 2a\alpha xu + b\alpha xv + b\alpha yv\\ &= \alpha \widetilde{B}((x, y), (u, v)).\end{aligned}$$

Remark 1.2. *If we set* $\lambda = 0$ *in* (1.1) *we get* $q(0) = 0$, *and if we set* $\lambda = -1$ *we obtain* $q(-x) = q(x)$, *that is* q *is an even function.*

Lemma 1.3. *For any bilinear and symmetric form* B, *there is an associated quadratic form given by* $q(x) = B(x, x)$.

The form B *is called the polar form of* q.

Proof. 1) $q(\lambda x) = B(\lambda x, \lambda x) = \lambda^2 B(x, x) = \lambda^2 q(x)$.

$$\begin{aligned} 2)\quad \widetilde{B}(x, y) &= q(x + y) - q(x) - q(y) = B(x + y, x + y) - B(x, x) - B(y, y) \\ &= B(x, y) + B(y, x) = 2B(x, y), \end{aligned} \tag{1.3}$$

which is also a bilinear and symmetric form. □

Example 1.10. *If we take* B *the inner product over* V, *then,* $q(x) = \langle x, x \rangle = \|x\|^2$.

1.2.1 Polarization identity

Proposition 1.3. *Let* q *be a quadratic form, then there exists a unique bilinear and symmetric form* B, *such that* $q(x) = B(x, x)$.

Proof. Indeed, let

$$B(x, y) = \frac{1}{2}\left[q(x + y) - q(x) - q(y)\right],$$

then, B is bilinear and symmetric and

$$B(x, x) = \frac{1}{2}\left[q(2x) - 2q(x)\right] = \frac{1}{2}\left[4q(x) - 2q(x)\right] = q(x).$$

Suppose that there exists an other bilinear and symmetric form B^* such that $q(x) = B^*(x, x)$, then,

$$q(x + y) - q(x) - q(y) = B^*(x + y, y + y) - B^*(x, x) - B^*(y, y) = 2B^*(x, y).$$

Therefore,

$$B^*(x, y) = B(x, y), \ \forall x, y \in V,$$

which shows the uniqueness of B. □

Definition 1.9. *The identity*

$$B(x, y) = \frac{1}{2}\left[q(x + y) - q(x) - q(y)\right], \tag{1.4}$$

is called polarization identity.

The bilinear form B *given by* (1.4), *is called the polar form associated to* q.

Remark 1.3. *The polar form associated to a quadratic form is alyaws symmetric. The underlying form* $\widetilde{B}$ *and the polor form* B *are related by the relation* $\widetilde{B} = 2B$.

Lemma 1.4. *A quadratic form q satisfies also the identities*

$$B(x,y) = \frac{1}{4}\left[q(x+y) - q(x-y)\right] \tag{1.5}$$

and

$$q(x+y) + q(x-y) = 2\left(q(x) + q(y)\right),$$

the last one is called parallelogram identity.

Proof. Let B be the polar form of q, from the identity (1.4) we have

$$q(x+y) = 2B(x,y) + q(x) + q(y)$$

then, replace y by $-y$ we get,

$$\begin{aligned} q(x-y) = q(x+(-y)) &= 2B(x,-y) + q(x) + q(-y) \\ &= -2B(x,y) + q(x) + q(y) \end{aligned}$$

consequently, (1.5) follows.
On the other hand

$$\begin{aligned} q(x+y) + q(x-y) &= B(x+y, x+y) + B(x-y, x-y) \\ &= 2B(x,x) + 2B(y,y) \\ &= 2\left(q(x) + q(y)\right), \end{aligned}$$

which proves the parallelogram identity. □

Definition 1.10. *Let q be a quadratic form and B be its polar form.*
1) We say that q is non degenerate if B is non degenerate. Any element $x \in V$, that satisfies $q(x) = 0$, is called isotropic.
2) We say that the quadratic form q is positive if

$$q(x) \geq 0,\ \forall x \in V,$$

3) the quadratic form q is called definite if it hasn't any non-zero isotropic element, that is

$$q(x) = 0 \Longleftrightarrow x = 0.$$

A positive definite quadratic form is such that

$$q(x) > 0,\ \forall x \neq 0.$$

1.2.2 Cauchy Schwarz and Minkowsky's inequalities

Proposition 1.4. *(**Cauchy Schwarz inequality**) Let B be a symmetric and positive bilinear form and let q be the underlying quadratic form of B, Then, for all $x, y \in V$, B and q satisfy the Cauchy-Schwarz inequality,*

$$|B(x,y)| \leq \sqrt{q(x)}\sqrt{q(y)}.$$

In addition, if B is definite, the equality is reached if and only if x and y are collinear ($y = \lambda x$).

Demonstration. *1) For any* $t \in \mathbb{R}$*, we have*

$$\begin{aligned} q(tx+y) &= 2B(tx,y) + q(tx) + q(y) \\ &= 2tB(x,y) + t^2 q(x) + q(y) \geq 0. \end{aligned}$$

If $q(x) = 0$*, then*

$$2tB(x,y) + q(y) \geq 0, \forall t \in \mathbb{R}$$

which entails that $B(x,y) = 0$*. If* $q(x) \neq 0$*, the trinomial* $P(t) = q(x)t^2 + 2B(x,y)t + q(y)$*, doesn't change sign, thus,* $\Delta = 4B^2(x,y) - 4q(x)q(y) \leq 0$*, and the result follows.*

2) On the other hand, if $y = \lambda x$*, we get*

$$B(x,\lambda x) = \lambda B(x,x) = \lambda q(x) = \sqrt{q(x)}\sqrt{\lambda^2 q(x)} = \sqrt{q(x)}\sqrt{q(y)}.$$

reciprocally, if the equality holds, the discriminant will be zero, and therefore $q(tx+y) = 0$*, and since* q *is definite there exists* t_0*, such that* $tx + y = 0$*, et* $y = -t_0 x$*.*

Corollary 1.1. *(**Minkowsky inequality**) If* q *is positive, then*

$$\forall x, y \in V : \sqrt{q(x,y)} \leq \sqrt{q(x)} + \sqrt{q(y)}.$$

Proof. Let B be the polar form of q, then

$$q(x+y) = q(x) + 2B(x,y) + q(y).$$

On the other hand, from the Cauchy-Schwarz inequality we have

$$\begin{aligned} 0 &\leq q(x+y) \leq q(x) + 2\sqrt{q(x)}\sqrt{q(y)} + q(y) \\ 0 &\leq q(x+y) \leq \left(\sqrt{q(x)} + \sqrt{q(y)}\right)^2 \end{aligned}$$

which completes the proof. □

Exercise Sheet 1

Exercise 1.1. *Let $V = \mathbb{R}[X]$ be the space of polynomials in x, and $a, b \in \mathbb{R}$. Define the form B by*

$$B(p, q) = p(a)\, q(b).$$

1) Show that B is a bilinear form on V.

2) Is it symmetric or skew-symmetric?

Exercise 1.2. *Let $V = C([a, b], \mathbb{R})$ and $B : V \times V \longrightarrow \mathbb{R}$, given by*

$$B(f, g) = \int_a^b f(x)\, g(x)\, dx.$$

Prove that B is a bilinear and symmetric form.

Exercise 1.3. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a square matrix and B the form defined on $\mathbb{R}^n \times \mathbb{R}^n$ by $B(u, v) = u^T A v$.*

1) Prove that B is bilinear form?

2) Say when B is symmetric and when it is skew-symmetric?

Exercise 1.4. *B is the form defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by*

$$B((a, b), (c, d)) = 2ac + 4ad - bc$$

Find the matrix A for which $B(u, v) = u^T A v$.

Exercise 1.5. *On $\mathcal{M}_{n\times n}(\mathbb{R}) \times \mathcal{M}_{n\times n}(\mathbb{R})$ we define B by $B(A, C) = tr\left(A^T C\right)$.*

Show that B is bilinear form.

Exercise 1.6. *Let $\mathbb{R}_n[X]$ be the space of polynomials of degree at most n. Define a form B by*

$$B(P, Q) = \int_0^1 t P(t)\, Q'(t)\, dt.$$

Prove that B is a bilinear form that is neither symmetric nor skew-symmetric.

Exercise 1.7. *On the space of square matrix we define the functionals*

$$q_1(A) = (tr(A))^2,\ q_2(A) = tr\left(A^T A\right).$$

Show that q_1 and q_2 are quadratic forms.

Exercise 1.8. *Prove that $q(x, y) = ax^2 + bxy$ is a quadratic form on $\mathbb{R}^2$.*

Exercise 1.9. *Let q be a quadratic form on a vector space V. Assume that q is definite.*

Prove that q is either positive definite or negative definite, that is q does not change sign.

Exercise 1.10. *Let q be a quadratic form and B its polar form. Prove the identity*

$$B(x,y) = \frac{1}{2}\left[q(x) + q(y) - q(x-y)\right].$$

Exercise 1.11. *Let q be a quadratic form on V and B its polar form. Suppose that B is non degenerate, and let $f : V \longrightarrow V$ be a bijective function that satisfies $f(0) = 0$ and*

$$q(f(x) - f(y)) = q(x-y), \; \forall x, y \in V.$$

Prove that $B(f(x), f(y)) = B(x,y)$ and that $B(f(\lambda x + y), z) = \lambda B(f(x), z) + B(f(y), z), \;\; \forall x, y, z \in V, \forall \lambda \in \mathbb{R}$.

Show that f is linear.

Exercise 1.12. *Let q be a quadratic form on a vectorial space V over $\mathbb{R}$. Let $r \in \mathbb{R}$, we say that q reprensents r, if there exists $v \in V$ such that $q(v) = r$. Thus, q is isotropic if q represents 0.*

Suppose that q is isortopic and B is nondegenrate. Show that q represents every $r \in \mathbb{R}$.

Chapter 2

Bounded operators on Hilbert spaces

2.1 Bounded linear Operators

In this paragraph, X and Y are two normed linear spaces defined on the same scalar field $\mathbb{K}$.

Definition 2.1. *A linear operator is a mapping $T : D(T) \subset X \longrightarrow Y$ that satisfies:*

$$T(\alpha x + y) = \alpha T(x) + T(y), \ \forall \alpha \in \mathbb{K}, \forall x, y \in D(T),$$

where $D(T)$ is a linear subspace of X, called the domain of T.

The image of x is denoted Tx and

$$R(T) := \{y \in Y; \exists x \in D(T), y = Tx\}$$

is called the range or the image of T.

The space of all linear operators form X into Y is denoted by $L(X,Y)$, it is a linear space over $\mathbb{K}$ with respect to the addition and multiplication by scalars,

$$(T+S)x = Tx + Sx, \ \forall x, y \in D(T) \cap D(S),$$
$$(\lambda T)x = \lambda(Tx), \ \forall x, y \in D(T).$$

The identity operator is denoted I, $I(x) = x, \forall x \in X$.

Definition 2.2. *The operator $T : D(T) \subset X \longrightarrow Y$ is said to be continuous or bounded if it satifies the property*

$$\forall x \in D(T), \forall \varepsilon > 0, \exists \delta > 0 : \forall y \in D(T); \|x - y\| < \delta \Longrightarrow \|Tx - Ty\| < \varepsilon.$$

The space of all bounded operators from X into Y is denoted by $\mathscr{L}(X,Y)$ or $\mathscr{B}(X,Y)$. In particular, if $Y = X$ the space is denoted by $\mathscr{L}(X)$ and if $Y = \mathbb{K}$, the space $\mathscr{L}(X, \mathbb{K})$ is written X' and called the dual space of X. The elements of X' are called linear forms or functionals.

Proposition 2.1. *Let $T : D(T) \subset X \longrightarrow Y$ be a linear operator, the following statements are equivalent,*

1) T is uniformly continuous on $D(T)$,
2) T is continuous on $D(T)$,
3) T is continuous in 0,
4) $\exists C > 0 : \forall x \in D(T), \|x\| \leq 1 \Longrightarrow \|Tx\| \leq C$,
5) $\exists C > 0, \|Tx\| \leq C\|x\|, \ \forall x \in D(T)$.

Proof. It is clear that 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3). 3) $\Longrightarrow$ 4) Assume that T is continuous at 0, then for $\varepsilon = 1$ we have

$$\exists \delta > 0 : \forall x \in E, \|x\| < \delta \Longrightarrow \|Tx\| < 1,$$

then, for any $x \in E$ such that $\|x\| < 1$ we have $\left\|\frac{\delta x}{2}\right\| < \delta$, consequently

$$\left\|T\left(\frac{\delta x}{2}\right)\right\| < 1,$$

therefore,

$$\|Tx\| < \frac{2}{\delta} = C.$$

To prove that 4) $\Longrightarrow$ 5), let $x \in E$ if$x \neq 0$, we have $\left\|\frac{x}{\|x\|}\right\| = 1$, hence, by 4) we deduce

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq C,$$

which gives $\|Tx\| \leq C\|x\|$. If $x = 0$ clearly $\|T(0)\| = C\|0\|$. Thus 5) is satisfied. Finaly, let us show that 5) $\Longrightarrow$ 1). Let $x, y \in E$ and take $\varepsilon > 0$, since,

$$\|Tx - Ty\| = \|T(x - y)\| \leq C\|x - y\|$$

then, so that $\|Tx - Ty\| < \varepsilon$, it suffices that $C\|x - y\| < \varepsilon$, which is satisfied if $\|x - y\| < \frac{\varepsilon}{C}$. Thus, $\delta = \frac{\varepsilon}{C}$ guaranties the result. □

2.1.1 Norm of an operator

Lemma 2.1. *Let X and Y be two normed linear spaces, then, $\|\cdot\| : \mathscr{L}(Y, X) \longrightarrow \mathbb{R}_+$ define by*

$$\|T\|_{\mathscr{L}(Y,X)} = \sup\{\|Tx\|; x \in D(T), \|x\| \leq 1\}$$

is a norm on $\mathscr{L}(Y, X)$.

Proof. i) Suppose that $T = 0$, that is $Tx = 0$ for all $x \in D(T)$, then, $\sup\limits_{x \in D(T), \|x\| \leq 1} \|Tx\| = 0$. Thus, $\|T\| = 0$.
Reciprocally, suppose that $\|T\| = \sup\limits_{\|x\| \leq 1} \|Tx\| = 0$, then, $Tx = 0, \forall x \in D(T), \ \|x\| \leq 1$.

Suppose that $x \in D(T)$, $\|x\| > 1$, then $z = \dfrac{x}{\|x|}$ is such that $\|z\| = 1$ consequently $Tz = 0$, which implies that $Tx = 0$ and therefore, $Tx = 0,\ \forall x \in D(T)$.
ii) Let $\lambda \in \mathbb{C}$, then,

$$\|\lambda T\| = \sup_{\|x\|\le 1} \|\lambda Tx\| = |\lambda| \sup_{\|x\|\le 1} \|Tx\| = |\lambda| \, \|T\| \, .$$

iii) Let $S, T \in \mathscr{L}(Y, X)$

$$\begin{aligned}\|T+S\| &= \sup\{\|Tx+Sx\|\,; x \in D(T)\cap D(S), \|x\| \le 1\} \\ &\le \sup\{\|Tx\| + \|Sx\|\,; x \in D(T)\cap D(S)\, \|x\| \le 1\} \text{ triangular inequality} \\ &\le \sup\{\|Tx\|\,; x \in x \in D(T), \|x\| \le 1\} + \sup\{\|Sx\|\,; x \in D(S), \|x\| \le 1\} = \|T\| + \|S\|,\end{aligned}$$

which completes the proof of the lemma. □

Example 2.1. *Let $T : C_{\mathbb{R}}[0,1] \longrightarrow \mathbb{R}$, be defined by $Tf = f(0)$, and equipped $C_{\mathbb{R}}[0,1]$ by the usual norm $\|f\| = \sup_{0\le x\le 1} |f(x)|$. We have*

$$\|T\| = \sup_{\|f\|\le 1} |f(0)| \le \sup_{\|f\|\le 1} \sup_{0\le x\le 1} |f(x)| = \sup_{\|f\|\le 1} \|f\| = 1.$$

On the other hand, let $g : [0,1] \longrightarrow \mathbb{R}$ with $g(x) = 1$, for all $x \in [0,1]$. Then, $g \in C[0,1]$ and $\|g\| = 1$. Moreover, $|Tg| = |g(0)| = 1$, hence, $\|T\| \ge |Tg| = 1$, therefore $\|T\| = 1$.

Proposition 2.2. *The norm defined above is also given by*

$$\|T\| = \inf\{C > 0 : \|Tx\| \le C\|x\|, \forall x \in D(T)\}.$$

Demonstration. *1) Since T is continuous, then from propsition 2.1, there exists $C > 0$, such that $\|Tx\| \le C\|x\|,\ \forall x \in D(T)$.*

For all $C \in \{C > 0 : \|Tx\| \le C\|x\|, \forall x \in D(T)\}$, we have

$$\|T\| = \sup_{x\in D(T), \|x\|\le 1} \|Tx\| \le \sup_{x\in D(T)} \|Tx\| \le C.$$

Thus,

$$\|T\| \le \inf\{C > 0 : \|Tx\| \le C\|x\|, \forall x \in D(T)\}.$$

Conversely, denote $\inf\{C > 0 : \|Tx\| \le C\|x\|, \forall x \in D(T)\} = \overline{C}$, *then, for all $\varepsilon > 0$, there exists $x_\varepsilon \in D(T)$, such that*

$$\|Tx_\varepsilon\| > \left(\overline{C} - \varepsilon\right) x_\varepsilon.$$

Clearly, $x_\varepsilon \neq 0$, then,

$$\forall \varepsilon > 0, \exists y_\varepsilon = \frac{x_\varepsilon}{\|x_\varepsilon\|}; \|Ty_\varepsilon\| > \left(\overline{C} - \varepsilon\right),$$

therefore,

$$\forall \varepsilon > 0, \exists y_\varepsilon = \frac{x_\varepsilon}{\|x_\varepsilon\|}; \|T\| = \sup_{\|y_\varepsilon\|\le 1} \|Ty_\varepsilon\| > \left(\overline{C} - \varepsilon\right).$$

Passing to the limit $\varepsilon \longrightarrow 0$, we infer that

$$\|T\| \ge \overline{C} = \inf\{C > 0 : \|Tx\| \le C\|x\|, \forall x \in D(T)\},$$

hence the equality $\|T\| = \overline{C}$.

2.1.2 Topology of $\mathscr{L}(X)$

Let X be a Banach space and $\mathscr{L}(X)$ the space of all linear and bounded operators on X, it is an algebra with respect the addition, the multiplication by scalars defined above and the multiplication

$$TS(x) = T(S(x)),\ x \in \{x \in D(S); Sx \in D(T)\}.$$

The space $\mathscr{L}(X)$ can be endowed by three types of topologies:

1) **Topology of uniform convergence**: This topology is induced by the norm

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$$

and it characterized by the following type of convergence

$$T_n \longrightarrow T \text{ uniformly in } \mathscr{L}(X) \Longleftrightarrow \lim_{n \longrightarrow \infty} \|T_n - T\| = 0.$$

$\mathscr{L}(X)$ is a Banach algebra with respect this topology.

2) **Strong topology**: This topology is characterized by the fact that a sequence of operators $(T_n)_{n \in \mathbb{N}}$ converges to the operator T, if

$$T_n \stackrel{s}{\to} T \Longleftrightarrow \lim_{n \longrightarrow \infty} \|T_n x - Tx\| = 0,\ \forall x \in D(T).$$

3) **Weak Topology**: it characterized by the following type of convergence

$$T_n \stackrel{*}{\rightharpoonup} T \Longleftrightarrow \lim_{n \longrightarrow \infty} \langle f, (T_n - T)x \rangle = 0, \forall x \in D(T), \forall f \in X'.$$

These three topologies are classified as follows:
The uniform topology is stronger that the strong topology which is stronger than the weak topology.

2.1.3 Closed operator

Definition 2.3. *The operator $A : D(A) \subset X \longrightarrow Y$, is said to be closed if $D(A) \times R(A)$ is closed in the space $X \times Y$; that is*

$$\forall (x_n) \subset D(A) : \lim x_n = x,\ \textit{then},\ x \in D(A),\ \textit{and}\ \lim Ax_n = Ax.$$

Remark 2.1. *Let $A : D(A) \subset X \longrightarrow Y$ be an operator. We sometimes endowed the domain $D(A)$ by the norm*

$$\|x\|_{D(A)} = \|x\|_X + \|Ax\|_Y.$$

Theorem 2.1. *(closed Graph Theorem) Let X, Y be Banach spaces and $A : D(A) \subset X \longrightarrow Y$ be a linear operator. If the graph $G(A)$ is closed in the topology of $D(A)$, then, the operator A is bounded.*

Demonstration. *Since $X \times Y$ is a Banach space and $G(A)$ is closed, then $G(A)$ is a Banach subspace of $X \times Y$.*
Define the linear transformation $R : G(A) \longrightarrow D(A)$ by $R(x, Ax) = x$. Then, R is a bijection between G and $D(A)$. Moreover

$$\|R(x, Ax)\| = \|x\| \le \|x\| + \|Ax\| = \|(x, Ax)\|_{\mathcal{G}(A).}$$

Therefore, R is bounded and $\|R\| \le 1$. Consequently, from the open mapping theorem, there exists $S : D(A) \longrightarrow G(A)$ such that $SR = I_{\mathcal{G}(A)}$ and $RS = I_{D(A)}$. In particular $Sx = (x, Ax)$, for all $x \in D(A)$.
Thus, $\|Ax\| \le \|x\| + \|Ax\| = \|(x, Ax)\| = \|Sx\| \le \|S\|\|x\|$, which shows that A is bounded.

Remark 2.2. *If A is a closed operator, then, $KerA$ is closed in X.*

2.1.4 Invertible operators

Definition 2.4. *An operator $T \in \mathscr{L}(X, Y)$ is said to be invertible if there exists an operator $S : R(T) \subset Y \longrightarrow X$, such that $S \in \mathscr{L}(Y, X)$ and $ST = I_{D(T)}$ and $TS = I_{R(T)}$. In this case S is denoted T^{-1}.*

Example 2.2. *For $f \in C[0,1]$ and defined $T_f \in L(L^2[0,1])$ by*

$$(T_f u)(x) = f(x) u(x), \; u \in L^2[0,1].$$

Clearly, $T_f \in \mathscr{L}(L^2[0,1])$. Let f be the function defined by $f(x) = 1 + x$. Then, T_f is invertible. Indeed, for $g(x) = \frac{1}{x+1}$, we have $T_g \in \mathscr{B}(L^2[0,1])$. Morevove,

$$(T_f T_g u)(x) = f(x) g(x) u(x) = u(x)$$

and

$$(T_g T_f u)(x) = g(x) f(x) u(x) = u(x)$$

which shows that T_f is invertible and $T_f^{-1} = T_g$.

Theorem 2.2. *Let X be a Banach space and $T \in \mathscr{L}(X)$ with $\|I - T\| < 1$, then, T is invertible with*

$$T^{-1} = \sum_{n \ge 0} (I - T)^n.$$

Proof. Since $\|I - T\| < 1$ the serie $\sum\limits_{n \ge 0} \|I - T\|^n$ converges. On the other hand $\|(I - T)^n\| \le \|I - T\|^n$, then the serie $\sum\limits_{n \ge 0} \|(I - T)^n\|$ converges and $\sum\limits_{n \ge 0} (I - T)^n$ is absolutely convergent serie, let S be its limit and $S_k = \sum\limits_{n=0}^{k} (I - T)^n$, then we have

$$\|TS_k - I\| = \|(I - (I - T)) S_k - I\| = \left\|(I - T)^{k+1}\right\| \le \|(I - T)\|^{k+1}.$$

Thus,

$$0 \leq \lim_{k \longrightarrow \infty} \|TS_k - I\| \leq \lim_{k \longrightarrow \infty} \|(I-T)\|^{k+1} = 0.$$

Therefore, $TS - I = \lim_{k \longrightarrow \infty} (TS_k - I) = 0$. Similarly $ST - I = \lim_{k \longrightarrow \infty} (S_k T - I) = 0$, which completes the proof. □

Theorem 2.3. *Let T be a linear operator from normed linear space X into normed linear space Y. Then, T^{-1} exists and is continuous, if and only if there $m > 0$, such that*

$$\|Tx\| \geq m \|x\|, \ \forall x \in X.$$

Definition 2.5. *Let X, Y be normed linear spaces. If an invertible operator $T \in \mathscr{L}(X, Y)$ exists then X, Y are isomorphic, and T is an isomorphism (between X and Y).*

Lemma 2.2. *If the normed linear spaces X, Y, are isomorphic, then:*

a) $\dim X < \infty$ *if and only if* $\dim Y < \infty$, *in which case* $\dim X = \dim Y$,

b) *X is separable if and only if Y is separable,*

c) *X is complete (i.e., Banach) if and only if Y is complete (i.e., Banach).*

Theorem 2.4. *Soient X et Y deux espaces de Banach, alors si $T \in \mathscr{L}(X, Y)$ est bijectif, il est inversible.*

2.2 Riesz représentation theorem

Let H be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, and $\varphi \in H'$ be bounded linear functional on H.

Definition 2.6. *Let H be a Hilbert space over IK and let $H' = \mathscr{L}(H, \mathbb{K})$ its dual. Denote by $\langle \cdot | \cdot \rangle$ the inner product over H, and $\langle \cdot, \cdot \rangle$ the duality pairing over $H' \times H$, where for any $\varphi \in H'$ and any $v \in H$, $\langle \varphi, v \rangle$ is the value of φ in v. The following is called the Riesz Representation Theorem:*

Theorem 2.5. *For every $\varphi \in H'$, there exists a unique $f \in H$, such that for every $v \in H$ we have*

$$\langle \varphi, v \rangle = \langle v | f \rangle \ \forall v \in \mathcal{H}.$$

Moreover,

$$\|\varphi\|_{\mathcal{H}'} = \|f\|_{\mathcal{H}}.$$

Demonstration. *Denote by*

$$N(\varphi) = \{v \in \mathcal{H}; \langle \varphi, v \rangle = 0\}$$

the nul subspace of φ. $N(\varphi)$ is a closed subspace of H.
If $\langle \varphi, v \rangle = 0$ for every $v \in H$, then it suffices to choose $f = 0$.

Assume that $\varphi \neq 0$, *then,* $N(\varphi) \neq H$, *consequently,* $(N(\varphi))^{\perp} \neq \{0\}$ *and* $H = N(\varphi) \oplus (N(\varphi))^{\perp}$ *and there exists* $z \in (N(\varphi))^{\perp}$ *such that* $\langle \varphi, z\rangle \neq 0$, *Clearly* $z \neq 0$ *and one can take*

$$\langle \varphi, z\rangle = 1.$$

For every $v \in H$ *we have*

$$\langle \varphi, v - \langle \varphi, v\rangle z\rangle = \langle \varphi, v\rangle - \langle \varphi, v\rangle \langle \varphi, z\rangle = 0,$$

Thus, $v - \langle \varphi, v\rangle z \in N(\varphi)$ *and since* $z \in (N(\varphi))^{\perp}$ *one gets*

$$\langle v - \langle \varphi, v\rangle z | \overline{z}\rangle = 0,$$

Set $f = \dfrac{\overline{z}}{\|z\|^2}$, *then*

$$\langle v|\overline{z}\rangle = \langle \varphi, v\rangle \langle z|\overline{z}\rangle = \langle \varphi, v\rangle \|z\|^2 .$$

Thus,

$$\langle \varphi, v\rangle = \frac{\langle v|\overline{z}\rangle}{\|z\|^2} = \left\langle v| \frac{\overline{z}}{\|z\|^2}\right\rangle = \langle v|f\rangle .$$

This completes the proof of the first statement.

On the other hand, let $\|v\| \leq 1$, *then,*

$$\|\varphi\|_{\mathcal{H}'} = \sup_{\|v\|\leq 1} |\langle \varphi, v\rangle| = \sup_{\|v\|\leq 1} |\langle f|v\rangle| \leq \sup_{\|v\|\leq 1} \|f\| \|v\| \leq \|f\|_{\mathcal{H}} .$$

Set $v = \frac{f}{\|f\|}$, *then* $\|v\| = 1$, *we have*

$$\|\varphi\| \geq |\langle \varphi, v\rangle| = \frac{|\langle \varphi, f\rangle|}{\|f\|} = \frac{\langle f|f\rangle}{\|f\|} = \|f\| ,$$

Thus, $\|\varphi\|_{\mathcal{H}'} = \|f\|_{\mathcal{H}}$.

Remark 2.3. *From the Riesz representation Theorem, any Hilbert space* H *is identified with its dual* H' *and the duality pairing* $\langle \cdot, \cdot\rangle$ *can be showed as an inner product over* H.

Theorem 2.6. *(Lax-Milgram) Let* $B : H \times H \longrightarrow IK$ *be a bilinear continuous and coercive form, then for any linear and continuous form* $L : H \longrightarrow K$, *there exists a unique* $u \in H$ *such that*

$$B(u, v) = Lv, \ \forall v \in \mathcal{H}.$$

Furthermore, if B *is symmetric,* u *is satisfied the property*

$$\frac{1}{2}B(u, u) - Lu = \min_{v\in H} \left\{ \frac{1}{2}B(v, v) - Lv \right\} .$$

Exercise Sheet 2

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Exercise 2.1. *Assume that* $\{c_n\}_{n\geq 1} \in \ell^\infty$ *and let* T *be the transformation defined by*

$$T : \ell^2 \longrightarrow \ell^2$$
$$\{x_n\} \longrightarrow \{c_n x_n\}.$$

1) Prove that T *is linear continuous operator and determine its norm.*

2) Suppose the $\inf\{|c_n|, n \geq 1\} > 0$. *Prove that* T *is bijective. Determine in this case* T^{-1} *and calculate its norm.*

3) Assume that one of the c_n *is zero. Show that* T *is neither injective nor surjective and that* $\overline{R(T)} \neq \ell^2$.

4) Suppose that $\forall n \geq 1, c_n \neq 0$, *but* $\inf\{|c_n|, n \geq 1\} = 0$. *Show that* T *is injective but not surjective and that* $\overline{R(T)} = \ell^2$.

Exercise 2.2. *Let* $\{c_n\}_{n\geq 1} \in \ell^\infty$ *and* $T : \ell^1 \longrightarrow \mathbb{R}$ *be defined by* $T(\{x_n\}) = \sum_{n\geq 1} c_n x_n$.

Show that T *is continuous and determine its norm.*

Exercise 2.3. *Let* $k : [a, b] \times [a, b] \longrightarrow \mathbb{R}$ *be a continuous function and* $A : C[a, b] \longrightarrow C[a, b]$ *the operator defined by*

$$(Af)(x) = \int_a^b k(x, y) f(y)\, dy.$$

1) Prove that $A \in \mathscr{L}(C[a, b])$.

2) Set $k(x, y) = \gamma \sin(x - y)$. *Show that if* $|\gamma| < 1$, *then for any* $g \in C([a, b])$ *there exists a unique* $f \in C([a, b])$ *such that*

$$f(x) = g(x) + \int_a^b k(x, y) f(y)\, dy.$$

Exercise 2.4. *Let* P *be the space of polynomials on* t *over* $[0, 1]$ *and* $A : P \longrightarrow P$ *be definied* $A(p) = p'$.

Show that A *is not continuous.*

Exercise 2.5. *Prove that the set of all invertible operator is an open subspece of* $\mathscr{L}(\mathcal{H}, \mathcal{K})$.

Exercise 2.6. *Prove that if* $A \in \mathscr{L}(\mathcal{H}, \mathcal{K})$ *is invertible, then, for any* $x \in H$, *one has* $\|Ax\| \geq \|A^{-1}\|^{-1} \|x\|$.

Exercise 2.7. *Suppose that* X *is a Banach space,* Y *is a normed space and* $T \in \mathscr{L}(X, Y)$.
Prove that if there exists $\alpha > 0$ *such that* $\|Tx\| \geq \alpha \|x\|$ *for all* $x \in X$, *then* $R(T)$ *is closed.*

2.3 The adjoint of an operator in a Hilbert space

Let $A : D(A) \subset X \longrightarrow Y$ be an unbounded operator with dense domain $\overline{D(A)} = X$.

Proposition 2.3. *Define the set*

$$D(A^*) : \{\psi \in Y' : \exists c > 0 \text{ such that } |\langle \psi, Ax \rangle| \leq c \|x\|_X, \forall x \in D(A)\}$$

Then, for all $\psi \in D(A^*)$ there exists a unique $\varphi \in X'$ such that

$$\langle \psi, Ax \rangle = \langle \varphi, x \rangle, \forall x \in D(A).$$

Demonstration. *$D(A^*)$ is a linear subspace of Y'. Let $f : D(A) \longrightarrow Y$ be define by $f(x) = \langle \psi, Ax \rangle$, it is clear that f is linear and*

$$|f(x)| = |\langle \psi, Ax \rangle| \leq c \|x\|_X.$$

From Hahn-Banach Theorem, we deduce that f can be prolonged linearly by a unique $\varphi : X \longrightarrow IR$, such that

$$|\varphi(x)| \leq c \|x\|, \forall x \in X,$$

consequently, $\varphi \in X'$ and since φ is the prolongement of f, we get

$$\langle \psi, Ax \rangle = \langle \varphi, x \rangle, \forall x \in D(A).$$

Definition 2.7. *The mapping*

$$\begin{aligned} A^* &: D(A^*) \subset Y' \longrightarrow X' \\ &: \psi \longrightarrow \varphi = A^*\psi, \end{aligned}$$

is called the adjoint operator of A and denoted A^*, it is a bounded operator that satisfies

$$\langle A^*\psi, x \rangle = \langle \psi, Ax \rangle, \forall \psi \in D(A^*), \forall x \in D(A).$$

Remark 2.4. *It is necessary that $D(A)$ be dense in X, to define A^* correctly. Indeed, suppose that there exist $\varphi_1, \varphi_2 \in X'$ such that $A^*\psi = \varphi_1$ et $A^*\psi = \varphi_2$, then,*

$$\langle A^*\psi, x \rangle = \langle \varphi_1, x \rangle = \langle \varphi_2, x \rangle, \forall x \in D(A)$$

therefore

$$\langle \varphi_1 - \varphi_2, x \rangle = 0, \forall x \in D(A)$$

which implies that $\varphi_1 - \varphi_2 = 0$, if and only if $D(A)$ is dense in X.

Remark 2.5. *$D(A^*)$ can be not dense in Y'.*

Theorem 2.7. *Let H and K be two Hilbert spaces over the same scalar field IK, and let $A \in \mathscr{L}(\mathcal{H}, \mathcal{K})$ be a linear and bounded operator. Then, there exists a unique operator from $A^* \in \mathscr{L}(\mathcal{K}, \mathcal{H})$ such that*

$$(Ax, y) = (x, A^*y), \forall x \in \mathcal{H}, \forall y \in \mathcal{K}.$$

Demonstration. *Fix* $y \in K$, *and let* $f : H \longrightarrow \mathbb{R}$ *be defined by* $f(x) = (Ax, y)$. *Clearly,* f *is linear, further we have*

$$|f(x)| = |(Ax, y)| \leq \|Ax\| \|y\| \leq \|A\| \|y\| \|x\| = C \|x\|,$$

which shows that f *is continuous. Therefore,* $f \in H'$. *On the other hand, using Riesz representation theorem, we infer that there exists a unique* $z \in H$ *such that*

$$(Ax, y) = f(x) = (x, z), \forall x \in \mathcal{H}.$$

Put $z = A^*y$, *we define a map* $A^* : K \longrightarrow H$ *which satisfies*

$$(Ax, y) = (x, A^*y), \forall x \in \mathcal{H}, \forall y \in \mathcal{K}.$$

It remains to show that $A^* \in \mathscr{L}(\mathcal{K}, \mathcal{H})$, *that is* A^* *is linear and continuous.*

First, for $y_1, y_2 \in K$ *and* $\alpha \in IK$, *we have for any* $x \in H$*:*

$$\begin{aligned}
(x, A^*(\alpha y_1 + y_2)) &= (Ax, \alpha y_1 + y_2) \\
&= \alpha (Ax, y_1) + (Ax, y_2) \\
&= \alpha (x, A^*y_1) + (x, A^*y_2) \\
&= (x, \alpha A^*y_1 + A^*y_2),
\end{aligned}$$

therefore, $A^*(\alpha y_1 + y_2) = \alpha A^*y_1 + A^*y_2$, *which shows the linearity of* A^*.

Secondly,

$$\|A^*y\|^2 = (A^*y, A^*y) = (AA^*y, y),$$

by Chauchy–Schwarz's inequality we deduce

$$\|A^*y\|^2 \leq \|AA^*y\| \|y\| \leq \|A\| \|A^*y\| \|y\|.$$

If $A^*y = 0$, *then,* $0 = \|A^*y\| \leq \|A\| \|y\|$.

If $A^*y \neq 0$, *dividing by* $\|A^*y\|$ *we obtain*

$$\|A^*y\| \leq \|A\| \|y\|, \text{ if } A^*y \neq 0,$$

then,

$$\|A^*y\| \leq \|A\| \|y\|, \forall y \in \mathcal{K}$$

which proves the boundedness of A^* *and that* $\|A^*\| \leq \|A\|$.

Finally, suppose that there exist two operators A_1^* *et* A_2^* *satisfy*

$$(Ax, y) = (x, A_1^*y) = (x, A_2^*y), \forall x \in \mathcal{H}, \forall y \in \mathcal{K},$$

then,

$$(x, (A_1^* - A_2^*) y) = 0, \forall x \in \mathcal{H}, \forall y \in \mathcal{K}$$

which implies that $(A_1^* - A_2^*) y = 0, \forall y \in K$, *hence* $A_1^* = A_2^*$ *and* A^* *is unique.*

Definition 2.8. *The operator* A^* *just constructed is called the adjoint operator of* A.

Example 2.3. *Endowed* $\mathbb{R}^2$ *by the canonical basis* $\{e_1, e_2\} = \{(1,0), (0,1)\}$ *and let* $A \in \mathscr{L}(\mathbb{R}^2)$ *given by*

$$A(x_1, x_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Assume that A^* *is given by a matrix* $B = (b_{ij})$, *that is*

$$A^*(y_1, y_2) = B \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Recall that

$$(Ax, y) = (x, A^*y),$$

that is,

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} b_{11}y_1 + b_{12}y_2 \\ b_{21}y_1 + b_{22}y_2 \end{pmatrix},$$

then,

$$a_{11}x_1y_1 + a_{12}x_2y_1 + a_{21}x_1y_2 + a_{22}x_2y_2 = b_{11}x_1y_1 + b_{12}x_1y_2 + b_{21}x_2y_1 + b_{22}x_2y_2$$

which gives,

$$b_{11} = a_{11}, b_{12} = a_{21}, b_{21} = a_{12}, b_{22} = a_{22}.$$

Thus,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^* = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}.$$

Remark 2.6. *Note that if* $A \in \mathscr{L}(\mathbb{C}^2)$, *then the adjoint of* A *is given by*

$$A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}.$$

Example 2.4. *Let* $k \in C[0,1]$ *and* $A \in \mathscr{L}(L^2[0,1])$ *defined by*

$$(Af)(x) = k(x) f(x).$$

We have

$$\langle Af, g \rangle = \langle f, A^*g \rangle, \forall f, g \in L^2[0,1],$$

that is

$$\int_0^1 k(x) f(x) g(x) \, dx = \int_0^1 f(x) A^*g(x) \, dx, \forall f, g \in L^2[0,1],$$

therefore, $A^*g(x) = k(x) g(x)$, *thus,* $A^* = A$.

Example 2.5. *Let* $T : \ell^2(\mathbb{R}) \longrightarrow \ell^2(\mathbb{R})$

$$T(x_n) = (0, x_2, x_3, \cdots)$$

$$\langle T(x_n), (y_n)\rangle = \sum_{n=2}^{\infty} x_n y_n$$

Now, suppose that $T^*(y_n) = (z_n)$, *then,*

$$\langle (x_n), T^*(y_n)\rangle = \langle (x_n), (z_n)\rangle = \sum_{n=1}^{\infty} x_n z_n = \sum_{n=2}^{\infty} x_n y_n.$$

For

Definition 2.9. *If* $A^* = A$ *we say that the operator* A *is self adjoint.*

Remark 2.7. *Clearly,* $I^* = I$.

Lemma 2.3. *Let* H, K *and* N *by three Hilbert spaces over* $\mathbb{C}$ *and let* $\lambda, \mu \in \mathbb{C}$, $A, B \in \mathscr{L}(\mathcal{H}, \mathcal{K})$ *and* $T \in \mathscr{L}(K, N)$. *Therefore,*

1) $(\lambda A + \mu B)^* = \overline{\lambda} A^* + \overline{\mu} B^*$,

2) $(AT)^* = T^* A^*$.

Proof. Exercise. □

Theorem 2.8. *Let* H, K *be two Hilbert spaces over* $\mathbb{C}$ *and* $A \in \mathscr{L}(\mathcal{H}, \mathcal{K})$, *then,*

1) $(A^*)^* = A$,

2) $\|A^*\| = \|A\|$,

3) *the function* $F : \mathscr{L}(\mathcal{H}, \mathcal{K}) \longrightarrow \mathscr{L}(\mathcal{K}, \mathcal{H})$ *defined by* $F(A) = A^*$ *is continuous,*

4) $\|A^* A\| = \|AA^*\| = \|A\|^2$.

Demonstration. *1) For the definition, we have* $(x, (A^*)^* y) = (A^* x, y) = \overline{(y, A^* x)} = \overline{(Ay, x)} = (x, Ay)$, $\forall x \in H$, $\forall y \in K$. *Thus,* $(A^*)^* = A$.

2) In the proof of Theorem 2.7, we have shown that $\|A^*\| \leq \|A\|$.

Applying this fact to A^* *we get* $\|A\| = \|(A^*)^*\| \leq \|A^*\|$ *then the equality followed.*

3) From the above lemma, we have

$$\|F(R) - F(S)\| = \|R^* - S^*\| = \|(R - S)^*\| = \|R - S\|,$$

then, for any $\varepsilon > 0$, *it suffices to take* $\delta = \varepsilon$, *and hence*

$$\forall \varepsilon > 0, \exists \delta > 0 : \|R - S\| < \delta \Longrightarrow \|R^* - S^*\| < \varepsilon.$$

4) Firstly, we have

$$\|A^* A\| \leq \|A^*\| \|A\| = \|A\|^2.$$

On the other hand

$$\|Ax\|^2 = (Ax, Ax) = (A^* Ax, x) \leq \|A^* Ax\| \|x\| \leq \|A^* A\| \|x\|^2,$$

therefore,

$$\|A\|^2 = \left(\sup_{\|x\| \leq 1} \|Ax\| \right)^2 = \sup_{\|x\| \leq 1} \|Ax\|^2 \leq \sup \|A^*A\| \|x\|^2 = \|A^*A\| .$$

Thus, $\|A^*A\| = \|A\|^2$.

Lemma 2.4. *Let* H, K *be two Hilbert spaces over* $\mathbb{C}$ *and* $A \in \mathscr{L}(\mathcal{H}, \mathcal{K})$, *then,*

1) $\ker A = (\operatorname{Im} A^*)^{\perp}$,

2) $\ker A^* = (\operatorname{Im} A)^{\perp}$,

3) $\ker A^* = \{0\}$ *if and only if* $\operatorname{Im} A$ *is dense in* K.

Proof. 1) Let $x \in \ker A$ and $z \in \operatorname{Im} A^*$ then $\exists y \in \mathcal{K}$ such that $z = A^*y$, we have

$$(x, z) = (x, A^*y) = (Ax, y) = (0, y) = 0,$$

this shows that $x \in (\operatorname{Im} A^*)^{\perp}$ and hence $\ker A \subset (\operatorname{Im} A^*)^{\perp}$. On the other hand, suppose that $x \in (\operatorname{Im} A^*)^{\perp}$. Since $A^*Ax \in \operatorname{Im} A^*$, then

$$\begin{aligned} &(x, A^*Ax) = 0, \\ &(x, A^*Ax) = (Ax, Ax) = \|Ax\|^2 = 0, \end{aligned}$$

therefore, $Ax = 0$, hence $x \in \ker A$ and $(\operatorname{Im} A^*)^{\perp} \subset \ker A$, consequently, $\ker A = (\operatorname{Im} A^*)^{\perp}$. 2) From 1) we deduce, $\ker A^* = (\operatorname{Im} (A^*)^*)^{\perp} = (\operatorname{Im} A)^{\perp}$. 3) Recall that $\left(F^{\perp}\right)^{\perp} = \overline{F}$ and if F is closed, $\left(F^{\perp}\right)^{\perp} = F$. Suppose that $\ker A^* = \{0\}$ then, from 2) we have $(\operatorname{Im} A)^{\perp} = \{0\}$, then $\left((\operatorname{Im} A)^{\perp}\right)^{\perp} = \{0\}^{\perp}$ which gives $\overline{\operatorname{Im} A} = \{0\}^{\perp} = \mathcal{K}$. Conversely, suppose that $\overline{\operatorname{Im} A} = \mathcal{K}$, that is, $\left((\operatorname{Im} A)^{\perp}\right)^{\perp} = \mathcal{K}$. Therefore

$$\left(\left((\operatorname{Im} A)^{\perp}\right)^{\perp}\right)^{\perp} = \mathcal{K}^{\perp} = \{0\} .$$

Since $(\operatorname{Im} A)^{\perp}$ is closed we have $\left(\left((\operatorname{Im} A)^{\perp}\right)^{\perp}\right)^{\perp} = (\operatorname{Im} A)^{\perp} = \ker A^*$. Consequently,

$$\ker A^* = \{0\}.$$

□

Corollary 2.1. *Let* H *be a* $\mathbb{C}$*–Hilbert space and* $A \in \mathscr{L}(\mathcal{H})$. *The following statements are equivalent*

Corollary 2.1. 1) *A is invertible,*

2 $\ker A^* = \{0\}$ *and there exists* $\alpha > 0$ *such that* $\|Ax\| \geq \alpha \|x\|, \forall x \in H$.

1) $\Longrightarrow$ 2) Suppose that A is invertible, then $\operatorname{Im} A = H$ and from number 3 of the previous lemma, $\ker A^* = \{0\}$. On the other hand, since A is invertible A^{-1} is bounded, then

$$\exists c > 0 : \left\|A^{-1}y\right\| \leq c \|y\|.$$

But $\operatorname{Im} A = H$, then

$$\forall x \in \mathcal{H}, \exists y \in \mathcal{H} : y = Ax, x = A^{-1}y.$$

Thus, for $\alpha = \dfrac{1}{c}$ we have

$$\left\|A^{-1}y\right\| \leq c \|y\| \Longleftrightarrow \alpha \|x\| \leq \|Ax\|.$$

2) $\Longrightarrow$ 1) If $\ker A^* = \{0\}$ then, $\operatorname{Im} A$ is dense in H. Let $y \in H$ and $\{y_n\} \subset \operatorname{Im} A$ be a sequence that converges to y. Then, $\{y_n\}$ is a Cauchy sequence and

$$\|y_n - y_m\| = \|Ax_n - Ax_m\| = \|A(x_n - x_m)\| \geq \alpha \|x_n - x_m\|,$$

therefore $\{x_n\}$ is a Cauchy too. Since H is complete, $\{x_n\}$ converges to an $x \in H$. Moreover, since A is continuous

$$y = \lim_{n\longrightarrow\infty} y_n = \lim_{n\longrightarrow\infty} Ax_n = A\left(\lim_{n\longrightarrow\infty} x_n\right) = Ax.$$

Consequently, $y \in \operatorname{Im} A$ and $\operatorname{Im} A$ is closed, which shows that $\operatorname{Im} A = H$. On the other hand, if $x \in \ker A$, one has $Ax = 0$, then $0 = \|Ax\| \geq \alpha \|x\| \geq 0$, which shows that A is injective, hence bijective. Since A is also continuous, we deduce by Banach theorem that A is invertible.

Example 2.6. *Let* $S \in \mathscr{L}(\ell^2)$ *defined by*

$$S(x_1, x_2, x_3 \cdots,) = (0, x_1, x_2, x_3, \cdots).$$

We can prove that $S^*(y_1, y_2, y_3, \cdots) = (y_2, y_3, \cdots)$. *Thus,* $(1, 0, 0, \cdots) \in \ker S^*$ *which shows that* $\ker S^* \neq \{0\}$. *Therefore,* S *is non invertible.*

Lemma 2.5. *Let* $A \in \mathscr{L}(\mathcal{H})$ *be a linear and continuous operator on the Hilbert space* H. *Then,* A *is invertible if and only if* A^* *est invertible. In this case, also we have* $(A^*)^{-1} = (A^{-1})^*$.

Suppose that A is invertible, then A^{-1} exists and we have $AA^{-1} = A^{-1}A = I$, then, $(AA^{-1})^* = (A^{-1}A)^* = I$, consequently,

$$\left(A^{-1}\right)^* A^* = A^* \left(A^{-1}\right)^* = I.$$

Thus, A^* is invertible et $(A^*)^{-1} = (A^{-1})^*$.

Reciprocally, if A^* is invertible, then, using the above argument, we deduce that $A = (A^*)^*$ is invertible and

$$A^{-1} = ((A^*)^*)^{-1} = \left((A^*)^{-1}\right)^*$$

consequently, $(A^{-1})^* = (A^*)^{-1}$.

2.4 Self adjoint and normal operators

We have already defined the self adjoint operator. Let us give the definition of normal operator

Definition 2.10. *let H be a Hilbert space and $A \in \mathscr{L}(\mathcal{H})$. The operator A is said to be normal if*

$$A^*A = AA^*.$$

Example 2.7. *Let $k \in C[0,1]$ and consider the complex Hilbert space $L^2[0,1]$. Let $A \in \mathscr{L}(L^2[0,1])$ be defined by*

$$(Af)(x) = k(x) f(x),$$

then

$$(A^*f)(x) = \overline{k(x)} f(x)$$

and

$$(A^*Af)(x) = (A^*kf)(x) = \overline{k(x)}k(x) f(x) = k(x)\overline{k(x)} f(x) = (AA^*f)(x).$$

Thus,

$$(A^*Af) = (AA^*f),$$

which shows that A is normal.

Lemma 2.6. *Let $S(\mathcal{H})$ be the set of all self adjoint operators over H. Then, for all $\lambda, \mu \in \mathbb{R}$, $\lambda S + \mu T \in S(\mathcal{H})$ and $S(\mathcal{H})$ is a closed linear subspace of $\mathscr{L}(\mathcal{H})$.*

Proof. 1) Let $\lambda, \mu \in \mathbb{R}$ and $S, T \in \mathcal{S}(\mathcal{H})$. We have

$$(\lambda S + \mu T)^* = \lambda S^* + \mu T^* = \lambda S + \mu T.$$

2) Moreover, let $\{T_n\}$ be a sequence of self–adjoint operators which converges to $T \in \mathscr{L}(\mathcal{H})$. since the function $T \longrightarrow T^*$ is continuous, then, $\{T_n^*\}$ converges to T^*. But $T_n^* = T_n$ then $\{T_n^*\}$ converges to T. By the uniqueness of the limit we have $T^* = T$. □

Lemma 2.7. *Let H be a complex Hilbert space and $T \in \mathscr{L}(\mathcal{H})$, then,*
1) TT^* et T^*T are self-adjoint operators,
2) there exist two self-adjoint operators R, S such that $T = R + iS$.

Proof. 1) $(TT^*)^* = (T^*)^* T^* = TT^*$ et $(T^*T)^* = T^*(T^*)^* = T^*T$. 2) Set $R = \dfrac{T + T^*}{2}$ and $S = \dfrac{T - T^*}{2i}$ then $T = R + iS$ et

$$R^* = \left(\frac{T + T^*}{2}\right)^* = \frac{T^* + T}{2} = R$$

and

$$S^* = \left(\frac{T - T^*}{2i}\right)^* = \frac{T^* - T}{-2i} = S.$$

□

Lemma 2.8. *Let H be a complex Hilbert space, $A \in \mathscr{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. Then, A is normal if and only if $(A - \lambda I)$ is normal.*

Proof. Si A est normal, alors

$$\begin{aligned}(A-\lambda I)^*(A-\lambda I) &= \left(A^* - \overline{\lambda} I\right)(A-\lambda I)\\ &= A^*A - \lambda A^* - \overline{\lambda} A + \overline{\lambda}\lambda I\\ &= AA^* - \overline{\lambda} A - \lambda A^* + \overline{\lambda}\lambda I\\ &= A\left(A^* - \overline{\lambda} I\right) - \lambda I\left(A^* - \overline{\lambda} I\right)\\ &= (A-\lambda I)\left(A^* - \overline{\lambda} I\right)\\ &= (A-\lambda I)(A-\lambda I)^*.\end{aligned}$$

Reciprocally, if $(A - \lambda I)$ is normal, then, $A = (A - \lambda I) - (-\lambda) I$ is normal. □

Lemma 2.9. *Let $A \in \mathscr{L}(\mathcal{H})$ be a normal operator. Then,*

1) $\|Ax\| = \|A^*x\|, \forall x \in H$.

2) *If there exists $\alpha > 0$ such that $\|Ax\| \geq \alpha \|x\|, \forall x \in H$, then* $\ker A^* = \{0\}$.

Proof. 1) Let $x \in \mathcal{H}$, since $A^*A = AA^*$, we have $A^*Ax = AA^*x$ and hence

$$\langle A^*Ax, x\rangle = \langle AA^*x, x\rangle$$

and

$$\langle Ax, Ax\rangle = \langle A^*x, A^*x\rangle \Longleftrightarrow \|Ax\|^2 = \|A^*x\|^2.$$

2) Let $y \in \ker A^*$, then, $A^*y = 0$ and by virtue of 1) one gets

$$0 = \|A^*y\| = \|Ay\| \geq \alpha \|y\|$$

therefore $\|y\| = 0$, which implies that $y = 0$. Thus, $\ker A^* = \{0\}$. □

Corollary 2.2. *Let $A \in \mathscr{L}(\mathcal{H})$ be a normal operator, the following statments are equivalent*

1) *A is invertible,*

2) *there exists $\alpha > 0$, such that $\|Ax\| \geq \alpha \|x\|, \forall x \in H$.*

Proof. From corollary 2.1 and number 2) in the previous lemma. □

Definition 2.11. *An operator $A \in \mathscr{L}(\mathcal{H})$ is said to be unitary if $A^*A = AA^* = I$.*

An isomory is an operator $A \in \mathscr{L}(\mathcal{H})$ that satisfies $\|Ax\| = \|x\|, \forall x \in H$, the norm of an isomerty is equal 1, that is $\|A\| = 1$.
We denote by $U(\mathcal{H})$ the set of all unitary operators over H.

Remark 2.8. *A unitary oprator is invertible and its inverse is its adjoint.*

Theorem 2.9. *Let $A, B \in \mathscr{L}(\mathcal{H})$, then,*
1) $A^*A = I$ if and only if A is isomerty.
2) B is unitary if and only if B is an isometry from H into H.

Demonstration. *1) Suppose that $A^*A = I$, then,*

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle x, A^*Ax\rangle = \langle x, x\rangle = \|x\|^2 .$$

Reciprocally, if A is an isometry then

$$\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 = \|x\|^2 = \langle x, x\rangle = \langle Ix, x\rangle, \forall x \in \mathcal{H}$$

therefore, $A^*A = I$.

2) Assume that B is a unitary operator, then, from 1) B is an isometry, and moreover, $\forall y \in H$, $y = B(B^*y) \in \mathrm{Im}\ H$, then $\mathrm{Im}\ H = H$.

Reciprocally, if B is an isometry from H into H, from 1) we deduce that, $B^*B = I$, and since $\mathrm{Im}\ H = H$ we have,

$$\forall y \in \mathcal{H},\ x \in \mathcal{H};\ y = Bx$$

therefore,

$$BB^*y = BB^*(Bx) = B(B^*Bx) = Bx = y$$

and consequently B is unitary.

Lemma 2.10. *Let X be a complex inner product space and $S, T \in (X)$. Then, $S = T$ if and only if $\langle Tx, x\rangle = \langle Sx, x\rangle$, $\forall x \in D(T) \cap D(S)$.*

Proof. Exercise. □

Lemma 2.11. *Let $U(H)$ be the set of all unitary operators over H, then*
1) If $A \in U$ then $A^* \in U$ and $\|A\| = \|A^*\| = 1$,
2) if $A, B \in U$, then, $AB \in U$ and $A^{-1} \in U$,
3) $U(\mathcal{H})$ is closed in $\mathscr{L}(\mathcal{H})$.

Proof. 1) Since $(A^*)^* = A$ and $A \in \mathcal{U}$, one gets

$$A^*A^{**} = A^{**}A^* = AA^* = I$$

which shows that $A^*$is unitary.
Moreover, $\|AA^*\| = \|A\|^2 = \|A^*\|^2 = \|I\| = 1$, then $= \|A\| = \|A^*\| = 1$.
2) $A^{-1} = A^*$, thus, $A^{-1} \in \mathcal{U}$.
Assume that $A, B \in \mathcal{U}$, then

$$(AB)(AB)^* = ABB^*A^* = A(BB^*)A^* = AIA^* = I.$$

3) Let (A_n) be a convergent sequence of unitary operators and let A be its limit. Since the function $T \to T^*$ is continuous, then $A_n^* \to A^*$. Further, we have

$$AA^* = \lim_{n\to\infty} (A_nA_n^*) = I,$$

and

$$A^*A = \lim_{n\to\infty} (\mathring{A}_n^*A_n) = I.$$

□

Exercise Sheet 3 .

Exercice 1. Let $T : \ell^2 \longrightarrow \ell^2$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, 4x_1, x_2, 4x_3, x_4, \cdots).$$

Determine T^* the adjoint of T.

Solution. Let $x = (x_n), y = (y_n) \in \ell^2$ and $z = (z_n) = T^*(y_n)$. From the definition we have

$$\langle Tx, y\rangle = \langle x, T^* y\rangle = \langle x, z\rangle,$$

that is,

$$4x_1\overline{y_2} + x_2\overline{y_3} + 4x_3\overline{y_4} + \cdots = x_1\overline{z_1} + x_2\overline{z_2} + x_3\overline{z_3} + \cdots$$

therefore,

$$x_1\overline{z_1} = 4x_1\overline{y_2}, x_2\overline{z_2} = x_2\overline{y_3}, x_3\overline{z_3} = 4x_3\overline{y_4}, \cdots$$

and

$$T^*(y_n) = (4y_2, y_3, 4y_4, \cdots)$$

Exercice 2. Let H be a Hilbert space with the inner product $\langle \cdot, \cdot\rangle$, $a, b \in H$ and $T, S \in \mathscr{L}(\mathcal{H})$ defined by $Tx = \langle a, b\rangle x$, $Sx = \langle x, a\rangle b$. Determine T^* and S^*.

Solution. Let $x, y \in H$ and $z = T^*y$, such that

$$\langle Tx, y\rangle = \langle x, T^* y\rangle = \langle x, z\rangle,$$

then,

$$\begin{aligned} \langle \langle a, b\rangle x, y\rangle &= \langle x, z\rangle \\ \langle x, z\rangle &= \langle a, b\rangle \langle x, y\rangle \\ &= \left\langle x, \overline{\langle a, b\rangle} y\right\rangle = \langle x, \langle b, a\rangle y\rangle, \forall x, y \in \mathcal{H}, \end{aligned}$$

therefore, $z = T^*y = \overline{\langle a, b\rangle} y = \langle b, a\rangle y$.

Let $w = S^*y$, then,

$$\begin{aligned} \langle Sx, y\rangle &= \langle x, S^* y\rangle = \langle x, w\rangle \\ \langle \langle x, a\rangle b, y\rangle &= \langle x, w\rangle, \ \forall x, y \in \mathcal{H}, \\ \langle x, a\rangle \langle b, y\rangle &= \langle x, w\rangle \\ \left\langle x, \overline{\langle b, y\rangle} a\right\rangle &= \langle x, w\rangle \end{aligned}$$

$$\langle x, w\rangle = \langle \langle x, a\rangle b, y\rangle = \langle x, a\rangle \langle b, y\rangle = \langle x, a\rangle \overline{\langle y, b\rangle} = \langle x, \langle y, b\rangle a\rangle.$$

Thus, $w = S^*y = \overline{\langle b, y\rangle} a = \langle y, b\rangle a$.

Exercice 3. Prove that $\ker T = \ker T^*T$.

Solution. Let $x \in \ker T$, then, $Tx = 0$ hence, $T^*Tx = 0$. Consequently, $x \in \ker T^*T$, then $\ker T \subset \ker T^*T$.

Reciprocally, if $x \in \ker T^*T$, then

$$0 = \langle x, T^*Tx \rangle = \langle Tx, Tx \rangle = \|Tx\|^2,$$

therefore, $Tx = 0$ and $x \in \ker T$. Thus, $\ker T^*T \subset \ker T$.

Exercice 4. Let $T : \ell^2 \longrightarrow \ell^2$ define by $T\{x_n\} = \{c_n x_n\}$, where $\{c_n\} \in \ell^\infty$.

Is T normal.

Solution. First, we have

$$\langle T\{x_n\}, \{y_n\} \rangle = \langle \{c_n x_n\}, \{y_n\} \rangle = \sum c_n x_n \overline{y_n} = \sum x_n c_n \overline{y_n} = \sum x_n \overline{\overline{c_n} y_n}$$

$$\langle T\{x_n\}, \{y_n\} \rangle = \langle \{x_n\}, T^*\{y_n\} \rangle = \sum x_n \overline{z_n}$$

therefore, $T^*\{y_n\} = \{\overline{c_n} y_n\}$.

On the other hand, we have

$$T^*T\{x_n\} = T^*\{c_n x_n\} = \{\overline{c_n} c_n x_n\}$$

and

$$TT^*\{x_n\} = T\{\overline{c_n} x_n\} = \{c_n \overline{c_n} x_n\}$$

that is $T^*T = TT^*$. Thus, T is normal.

Exercice 5. Let $T \in \mathscr{L}(\mathcal{H})$ such that $\|T^*x\| = \|Tx\|$, $\forall x \in H$. Prove that T is normal.

Solution.

$$\langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = \|Tx\|^2 = \|T^*x\|^2 = \langle T^*x, T^*x \rangle = \langle TT^*x, x \rangle$$

Therefore, $T^*T = TT^*$.

Chapter 3

Compact operators

3.1 Introduction

Let $(X, \mathcal{T})$ be a topological space. A subset $K \subset X$, is said to be compact if every open cover $\{U_i\}_{i\in I}$ of K has a finite subcover $\{U_i\}_{i\in I_0}$.

Lemma 3.1. *Let (X, d) be a metric space and $K \subset X$. Then, K is compact if and only if every sequence $(u_n) \subset K$ has a convergent subsequence (u_{n_k}) with limit $\ell \in K$.*

Definition 3.1. *A set K is said to be relatively compact if $\overline{K}$ is compact.*

Lemma 3.2. *Let X be an infinite dimensional normed linear space, then, the unit ball*

$$B := \{x \in X : \|x\| \leq 1\}$$

is never compact.

Let $x_0 \in X$ and $Span\{x_0\}$ be the subspace generates by x_0. Then, $Span\{x_0\}$ is finite dimensional subspace of X, consequently, $Span\{x_0\}$ is closed and $Span\{x_0\} \neq X$.

From Riesz's Lemma, we deduce that there exists $x_1 \in X$, $\|x_1\| = 1$ such that

$$\|x_1 - \alpha x_0\| > \frac{3}{4}, \quad \forall \alpha \in \mathbb{R}.$$

Similarly, $Span\{x_0, x_1\}$ is closed finite dimensional subspace of X, then $Span\{x_0, x_1\} \neq X$ and there exists $x_2 \in X$, $\|x_2\| = 1$ such that

$$\|x_2 - \alpha x_0 - \beta x_0\| > \frac{3}{4}, \ \forall x \in X, \ \forall \alpha, \beta \in \mathbb{R}.$$

We continue in the same way, we construct a unitary sequence $\{x_n\}$ that satisfies $\|x_n - x_m\| > \dfrac{3}{4}$, for all $n \neq m$. Therefore, we can't extract any convergent sequence from $\{x_n\}$ which shows that B is not compact.

3.2 Compact operators

Definition 3.2. *Let X, Y be normed spaces. A linear operator $T \in L(X,Y)$ is said to be compact if the image by T of every bounded set B of X is relatively compact in Y. The set of all compact operators from X into Y is denoted by $\mathscr{K}(X,Y)$.*

Proposition 3.1. *Let X, Y be normed spaces and $T \in \mathscr{L}(X,Y)$. The following statements are equivalent:*

1 *T is compact.*

2) *The image of the unit ball $B_X(0,1)$ of X is relatively compact in Y.*

3) *Every bounded sequence $\{x_n\}$ in X has a subsequence x_{n_k} such that $\{Tx_{n_k}\}$ converges in Y.*

Demonstration. *Clearly 1)$\Longrightarrow$2).*

2)$\Longrightarrow$3) Suppose that 2) holds, then there exists $r > 0$, such that $\{x_n\} \subset B_X(0,r) = rB_X(0,1)$, therefore, $\overline{T\{x_n\}} \subset \overline{rT(B_X(0,1))}$ which is compact, since $\overline{T\{x_n\}}$ is closed, then compact, consequently $T\{x_n\}$ is relatively compact.

3)$\Longrightarrow$1) Let $B \subset X$ be a bounded subset of X and $\{y_n\}$ a sequence of $\overline{T(B)}$. For all $n \in \mathbb{N}^*$, there exists $z_n \in T(B)$ such that $\|y_n - z_n\| < \frac{1}{2^n}$, consequently, there exists $x_n \in B$ such that $z_n = Tx_n$. Since $\{x_n\} \subset B$ is bounded, $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$. Thus

$$\lim_{n_k \longrightarrow \infty} \|y_{n_k} - z_{n_k}\| = 0,$$

consequently, $\{y_{n_k}\}$ converges to the same limit as $\{z_{n_k}\}$, which shows that $T(B)$ is relatively compact.

Lemma 3.3. *Every compact operator $T \in \mathscr{K}(X,Y)$ is continuous.*

Proof. Suppose that T is not continuous, then, there exists a sequence $\{x_n\}$ of unit vectors such that $\|Tx_n\| \geq n$, for all n. Since T is compact, one can extract a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ converges to $y \in Y$, but this contradicts the fact that $\|Tx_{n_k}\| \geq n_k$. Consequently T is continuous (bounded) and $\mathscr{K}(X,Y) \subset \mathscr{L}(X,Y)$. □

Theorem 3.1. *The set $\mathscr{K}(X,Y)$ is a closed subspace of $\mathscr{L}(X,Y)$ for the operator norm.*

Demonstration. *1) Let $S, T \in \mathscr{K}(X,Y)$ and $\alpha, \beta \in \mathbb{C}$. Let $\{x_n\}$ be a bounded sequence on X. Since S is compact, there exists a subsequence $\{x_{n_k}\}$ such that $\{Sx_{n_k}\}$ convergents. On the other hand, since T is compact, there exists a subsequence $\{x_{n_{k_m}}\}$ for which $\{Tx_{n_{k_m}}\}$ converges. Thus, $\{\alpha Sx_{n_{k_m}} + \beta Tx_{n_{k_m}}\}$ converges. Therefore $\alpha S + \beta T$ is compact.*

2) Let $\{T_n\}$ be a sequence of compact operators that converges to T, $\lim_{n \longrightarrow \infty} \|T_n - T\| = 0$. We will show that T is compact˙ For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N; \|T_n - T\| < \frac{\varepsilon}{2}.$$

Therefore,

$$\forall n \geq N, \forall x \in B_X(0,1); \|T_n x - Tx\| < \frac{\varepsilon}{2}.$$

Take $n \geq N$ *and let* $\{B_o(y_i, \varepsilon)\}_{i\in I}$ *where* $y_i \in Y$*, be an open cover of* $T(B_X(0,1))$*, then,* $\left\{B_o\left(y_i, \frac{3\varepsilon}{2}\right)\right\}_{i\in I}$ *is an open cover of* $T_n(B_X(0,1))$. *Since* T_n *is compact, there exists a finite subrecover* $\left\{B_o\left(y_i, \frac{3\varepsilon}{2}\right)\right\}_{i\in I_0}$ *of* $T_n(B_X(0,1))$, *consequently,* $\{B_o(y_i, 2\varepsilon)\}_{i\in I_0}$ *recover* $T(B_X(0,1))$. *Thus,* T *is compact.*

Proposition 3.2. *Let* $T \in \mathscr{L}(X,Y)$ *and* $S \in \mathscr{L}(Y,Z)$*. If at least one of the operators* S, T *is compact then* ST *is compact.*

Proof. Let $\{x_n\}$ be a bounded sequence in X. If T is compact then, there exists a subsequence on $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ converges in Y, and, since S is continuous, the sequence $\{STx_{n_k}\}$ still convergent.
If T is not compact, then $\{Tx_n\}$ still bounded and, since S is compact there exist a subsequence $\{STx_{n_k}\}$ such that $\{STx_{n_k}\}$ converges, therefore ST is compact. □

Definition 3.3. *An operator* T *is said to be of finite rank, if its range (image)* $R(T)$ *is of finite dimension. In this case we note* $dim R(T) = r(T)$.

Proposition 3.3. *1) An operator of finite rank is compact.*
2) If $\dim X$ *or* $\dim Y$ *is finite, then* $\mathscr{L}(X,Y) = \mathscr{K}(X,Y)$.

Proof. 1) Let $\{x_n\}$ be a bounded sequence in X, then $\{Tx_n\}$ is bounded in $R(T)$. Since $\dim R(T) < \infty$, from Bolzano-Weistrass theorem, we deduce that $\{Tx_n\}$ has a convergent subsequence. Thus T is compact.
2) If $\dim Y < \infty$, then $\dim R(T) \leq \dim Y < \infty$. If $\dim X < \infty$, then $\dim R(T) \leq \dim X$, and the result follows from the statement 1). □

Theorem 3.2. *Let* $\{T_n\} \subset \mathscr{L}(X,Y)$ *be a sequence of bounded operators of finite range and let* $T \in \mathscr{L}(X,Y)$ *be its limit, then* T *is compact.*

Proof. Since every operator of finite range is compact, and the set $\mathscr{K}(X,Y)$ is closed, T est compact. □

Example 3.1. *Let* $T \in \mathscr{L}(\ell^2)$ *be defined by:* $T\{x_n\} = \{n^{-1}x_n\}$.
For all $k \in N^*$ *define* $T_k\{x_n\} = \{y_n^k\}$ *such that*

$$y_n^k = \begin{cases} n^{-1}x_n, & 1 \leq n \leq k, \\ 0, & n > k. \end{cases}$$

Every operator T_k *is linear bounded and of finite range, thus compact. On the other hand,*

$$\lim_{k\to\infty} \|(T - T_k)\{x_n\}\|^2 = \lim_{k\to\infty} \sum_{n\geq k+1} \frac{|x_n|^2}{n^2} \leq \lim_{k\to\infty} \frac{1}{(k+1)^2} \sum |x_n|^2 \leq \lim_{k\to\infty} \frac{\|\{x_n\}\|}{(k+1)^2} = 0.$$

Therefore, $\lim\limits_{n\to\infty} T_k = T$ *which shows that* T *is compact.*

Remark 3.1. *In general, the converse is not true if Y is only a Banach space. However, if Y is a Hilbert space, the converse is also valid.*

Theorem 3.3. *Let X be a linear normed space, H a Hilbert space and $T \in \mathscr{K}(X, H)$ a compact operator. Then, there exists a sequence $\{T_n\}$ of finite rank operators which converges to T in $\mathscr{L}(X, H)$.*

Demonstration. *1) If T itself is of finite rank, there is no thing to prove.*
2) Suppose that T is not of finite rank. Then, $\overline{R(T)}$ is a closed separable subspace of H. Consequently, it is a separable Hilbert subspace. Let $\{e_n\}$ be an orthonormal basis of $\overline{R(T)}$.
For each $k \geq 1$ let $M_k := Span\{e_1, e_2, ..., e_k\}$ and P_k the orthogonal projection of $\overline{R(T)}$ on M_k and $T_k = P_k T$. Since $R(T_k) \subset M_k$, T_k is of finite rank.
We will prove that $\lim\limits_{k\to\infty} \|T_k - T\| = 0$. Suppose that this does not hold. Thus, we can extract a subsequence $\{T_{k_l}\}$ and there exists $\varepsilon > 0$ such that $\|T_{k_l} - T\| \geq \varepsilon$, for all $k_l \in N^*$. Therefore, there exists a sequence $\{x_{k_l}\} \subset X$ of unit vectors such that

$$\|(T_{k_l} - T)x_k\| \geq \frac{\varepsilon}{2}, \ \forall k_l \in \mathbb{N}^*.$$

Since T is compact, one can extract from $\{x_{k_l}\}$ a subsequence which we still denoted by $\{x_{k_l}\}$ such that $\{Tx_{k_l}\}$ converges to $y \in H$. We have

$$(T_{k_l} - T)x_{k_l} = (P_{k_l} - I)Tx_{k_l} = (P_{k_l} - I)y + (P_{k_l} - I)(Tx_{k_l} - y)$$

$$= -\sum_{n=k_l+1}^{\infty} (y, e_n)e_n + (P_{k_l} - I)(Tx_{k_l} - y),$$

consequently,

$$\frac{\varepsilon}{2} \leq \|(T_{k_l} - T)x_{k_l}\| + (\sum_{n=k_l+1}^{\infty} (y, e_n)^2)^{1/2} + (\|P_{k_l}\| + 1)\|Tx_{k_l} - y\| \to 0,$$

which is impossible. Thus, $\lim\limits_{k\to\infty} \|T_k - T\| = 0$.

Lemma 3.4. *Let X be linear normed space of infinite dimensional, then the identity operator is never compact.*

Proof. Since $\dim X = \infty$, from the lemma 3.2, there exists a sequence $\{x_n\} \subset X$ of unit vectors which has no convergent subsequence, then $\{Ix_n\} = \{x_n\}$ doesn't have any convergent subsequence. □

Corollary 3.1. *If X is an infinite dimensional linear normed space and $T \in K(X)$ a compact operator, then T is not invertible.*

Proof. Suppose that T is invertible, then $I = T^{-1}T$ is compact, which contradicts the lemma 3.4. □

Theorem 3.4. *Let $T \in \mathscr{K}(X,Y)$ then $R(T)$ and $\overline{R(T)}$ are separable.*

Demonstration. *It is well known that a compact subset of a metric space is separable and that a subset of a separable set is also separable.*
For every $n \in N^*$ we set $R_n = T(B(0,n))$ the image by T of the ball of radius n. Since T is compact R_n is relatively compact, then separable. Consequently, $R(T) = \underset{n\geq 1}{\cup} R_n$ is separable. On the other hand, every dense subset in $R(T)$ is also dense in $\overline{R(T)}$, hence $\overline{R(T)}$ is also separable.

3.3 The adjoint of compact operator

Lemma 3.5. *Let H be a Hilbert space and let $T \in \mathscr{L}(H)$ be an operator. Then, $r(T) = r(T^*)$. (Both finite or infinite dimensional).*

Demonstration. *Suppose that $r(T) < \infty$. For every $y \in H$ we write the orthogonal decomposition of y with respect to $\ker(T^*)$, $y = u + v$ where $u \in \ker(T^*)$ and $v \in (\ker(T^*))^\perp = \overline{R(T)}$. Since $r(T) < \infty$ we have $\overline{R(T)} = R(T)$. Thus, $T^*y = T^*(u+v) = T^*u + T^*v = T^*v$. Consequently, $R(T^*) = T^*(R(T))$ which implies that $r(T^*) \leq r(T) < \infty$.*
Applying this result for T^* and recalling that $(T^*)^* = T$ we conclude that $r(T) \leq r(T^*) < \infty$, thus the equality follows.
If one of $r(T)$ or $r(T^*)$ is infinite the other can't be finite.

Theorem 3.5. *Let H be a Hilbert space and $T\mathscr{L}(H)$, then T is compact if and only if T^* is compact.*

Proof. Suppose that T is compact, then, there exists a sequence of finite rank operators that converges to T. The adjoint T_n^* de chaque opérateur ests of any T_n is of finite rank because of $r(T^*) = r(T)$; On the other hand

$$\lim_{n\to\infty} \|T_n^* - T^*\| = \lim_{n\to\infty} \|T_n - T\| = 0$$

therefore T^* is compact as limit of a sequence of finite rank operators. □

Exercise Sheet 4
Master I

Exercise 1.
Recall that an orthogonal projection in a Hilbert space H is an operator P such that $P = P^* = P^2$.
An operator $T \in \mathcal{H}$ is said to be positive if $\langle Tx, x\rangle \geq 0,\ \forall x \in H$.
Let $P : C^3 \longrightarrow C^3$ by defined by

$$P(x, y, z) = (x, y, 0).$$

Prove that P is a positive projection.
Exercise 2.
Let M be a closed subset of a Hilbert space H. For any $x \in H$ let $x = u + v$ be the orthogonal decomposition with respect to M of x, that is $u \in M$, $v \in M^{\perp}$.
Let $P : H \longrightarrow H$ by defined by $Px = u$.
Prove that $P \in \mathscr{L}(H)$, P is projection, $\|P\| \leq 1$ and that $R(P) = M$ and $ker(P) = M^{\perp}$.
Exercise 3.
Let H be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$, and let $a, b \in H$. Define $T \in \mathscr{L}(H)$ by $Tx = \langle x, a\rangle b$. Show that T is compact.
Exercise 4.
Let H be a Hilbert space and $T \in \mathscr{L}(H)$. Prove that if T^*T is compact, then T and T^* are compact.
Exercise 5.
Let H be a Hilbert space and $(e_k)_{k\geq 1}$ an orthonormal basis of H. Define an operator T by

$$T\left(\sum_{k\geq 1} x_k e_k\right) = \sum_{k=2}^{\infty} \frac{1}{k} x_k e_{k-1}.$$

Show that T is compact and determine T^*.
Exercise 6.
Let k be a continuous function $k : [0,1] \times [0,1] \longrightarrow R$. Define the operator $T \in \mathscr{L}\left(C([0,1])\right)$ by

$$Tf(t) := \int_0^1 k(t,s)f(s)ds.$$

1) Show that for any $f \in C\left([0,1]\right)$ we have $Tf \in C\left([0,1]\right)$ and that $\|T\| = \sup_{t\in[0,1]} \int_0^1 |k(t,s)|ds$.
2) Show that T is a compact operator.

Solution

Exercise 1.
Clearly, P is linear. Moreover, since C^3 is finite dimensional, P is continuous. Or $\|P(x,y,z)\|^2 = |x|^2+|y|^2 \leq |x|^2+|y|^2+|z|^2$, which shows that $\|PU\| \leq \|U\|$, therefore, P is continuous.
On the other hand,

$$\langle P(x,y,z),(u,v,w)\rangle = x\overline{u} + y\overline{v} = \langle (x,y,z), P(u,v,w)\rangle.$$

Thus, $P^* = P$ (P is self-adjoint). Moreover, $P^2(x,y,z) = P(x,y,0) = (x,y,0) = P(x,y,z)$ which shows that $P^2 = P$.

Exercise 2.
First, we prove that P is linear and continuous.
Let $x, y \in H$ and $x = u + v$, $y = z + w$ then for all $\alpha \in C$ we have $\alpha x + y = (\alpha u + z) + (\alpha v + w)$. Since M is a subspace $\alpha u + z \in M$ and $\alpha v + w \in M^\perp$, therefore $P(\alpha x + y) = \alpha u + z = \alpha Px + Py$. Thus P is linear. Moreover, we have

$$\|x\|^2 = \langle u+v, u+v\rangle = \langle u, u\rangle = \|u\|^2,$$

then, $\|Px\|^2 = \|u\|^2 = \|x\|^2$, which shows that, P is continuous.
Next, we prove that P is an orthogonal projection.

$$\langle Px, y\rangle = \langle u, z+w\rangle = \langle u, z\rangle$$

and

$$\langle x, Py\rangle = \langle u+v, z\rangle = \langle u, z\rangle$$

Thus, $P^* = P$.
Finally, since $u \in M$ then $u = u + 0$ and $Pu = u$, therefore, $P^2x = PPx = Pu = u = Px$.

Clearly,$P(H) \subset M$. Moreover, if $x \in M$ then $Px = x$, then,$M \subset P(M)$, which shows that $M = R(P)$. On the other hand, since $ker(P) = (Im(P^*))^\perp = (Im(P))^\perp = M^\perp$.

Exercise 6. Let $f \in C([0,1])$ and set $\|f\|_1 = \int_0^1 |f(x)|dx$, then

$$\|f\|_1 = \int_0^1 |f(x)|dx \leq \|f\|_\infty \int_0^1 dx = \|f\|_\infty < \infty.$$

Next, let $\varepsilon > 0$. Since k is a continuous function on the compact set $[0,1]\times[0,1]$, it is actually uniformly continuous. Thus, we can choose $\delta > 0$ such that whenever $|\,(t,s)-(t',s')\,| < \delta$,

Chapter 4

Spectrum of an operator

4.1 Spectrum of a bounded operator

Definition 4.1. *Let H be a complex Hilbert space and $T \in \mathscr{L}(H)$ a bounded operator. The set of complex numbers λ such that the operator $T - \lambda I$ is invertible is called the resolvent set of T and denoted $\rho(T)$,*

$$\rho(T) := \{\lambda \in \mathbb{C};\ T - \lambda I \ \textit{est inversible}\}.$$

The elements of $\rho(T)$ are called regular points of T and for any $\lambda \in \rho(T)$ the operator $(T - \lambda I)^{-1}$ is bounded and is called the revolent operator and denoted $R(\lambda, T)$ at the point λ.

The spectrum of de T is the complement set of $\rho(T)$ and it denoted $\sigma(T)$

$$\sigma(T) := \mathbb{C} - \rho(T) = \{\lambda \in \mathbb{C};\ T - \lambda I \ \textit{n'est pas inversible}\}.$$

Example 4.1. *Let $\mu \in C$ and $T = \mu I$. We have $T - \lambda I = (\mu - \lambda) I$. Thus, $T - \lambda I$ is invertible if and only if $\lambda \neq \mu$, then $\sigma(T) = \{\mu\}$ et $\rho(T) = C - \{\mu\}$.*

Definition 4.2. *Let H be a complex Hilbert space and $T \in \mathscr{L}(H)$ an operator. A complex number $\lambda \in C$ is called an eigenvalue of T, if there exists $x \in H$, $x \neq 0$ such that $Tx = \lambda x$. Such an x is called eigenvector associated to λ. The set of all eigenvalues of T is denoted $VP(T)$.*

Lemma 4.1. *Each eigenvalue of T belongs to the spectrum of T, that is $VP(T) \subset \sigma(T)$.*

Proof. Suppose that $\lambda \in VP(T)$. Since there exists $x \neq 0$ such that $Tx = \lambda x$, then, $x \in \ker(T - \lambda I)$ consequently, $\ker(T - \lambda I) \neq \{0\}$, and then $(T - \lambda I)$ is not invertible. □

Lemma 4.2. *Suppose that H is finite dimensional and $T \in \mathscr{L}(H)$. Thus $\sigma(T) = VP(T)$.*

Proof. It suffices to prove that $\sigma(T) \subset VP(T)$.
In finite dimensional case, we have $\dim \mathcal{H} = \dim (R(T)) + \dim \ker (T)$. Let $\lambda \in \sigma(T)$ then, $(T - \lambda I)$ is not invertible. So $T - \lambda I$ is either non-injective then $\ker (T - \lambda I) \neq \{0\}$, or non surjective, then $\dim (R(T)) \neq \dim \mathcal{H}$. Consequently, $\ker (T - \lambda I) \neq \{0\}$. Thus, there exists $0 \neq x \in \ker (T - \lambda I)$ therefore, $Tx = \lambda x$ and $\lambda \in VP(T)$. □

Remark 4.1. *In the case when* $\dim H = +\infty$, *it will be exists* $\lambda \in \sigma(T)$ *which is not an eigenvalue.*

Example 4.2. *Let* $S \in \mathscr{L}(\ell^2)$ *be defined by*

$$S(x_n) = (0, x_1, x_2, \cdots).$$

S is not invertible, because $R(S) \neq \ell^2$. *Thus,* $0 \in \sigma(S)$. *But* 0 *can not be an eigenvalue of* T, *because there is no* $x \neq 0$ *that satisfies* $Sx = 0x$.

Theorem 4.1. *Let* H *be a Hilbert space and* $T \in \mathscr{L}(H)$, *then,*

1) *If* $|\lambda| > \|T\|$, $\lambda \notin \sigma(T)$,

2) $\sigma(T)$ *is closed in* C.

Demonstration. *1) If* $|\lambda| > \|T\|$, *then* $\|\lambda^{-1}T\| < 1$ *consequently,* $I - \lambda^{-1}T$ *is invertible, and* $T - \lambda I = -\frac{1}{\lambda}(T - \lambda I)$ *is also invertible, then* $\lambda \notin \sigma(T)$.

2) Define $F : C \longrightarrow \mathscr{L}(H)$ *by* $F(\lambda) = T - \lambda I$. *We have* $\|F(\lambda) - F(\mu)\| = \|(\mu - \lambda) I\| = |\lambda - \mu|$, *then* F *is continuous (it suffices to choose* $\alpha = \varepsilon$ *in the definition of continuity). Let* C *be the set of all non-invertible operators. It is a closed set because the set of all invertible operators is open. Consequently,* $\sigma(T) = F^{-1}(\mathcal{C})$ *is closed.*

Remark 4.2. *The spectrum of an operator* T *is a closed bounded set, then compact set included in* C. *It is in the circle of center at the origin and radius* $\|T\|$.

Lemma 4.3. *Let* $T \in \mathscr{L}(H)$, *then*

$$\rho(T^*) = \left\{\overline{\lambda} \in \mathbb{C} : \lambda \in \rho(T)\right\},$$
$$\sigma(T^*) = \left\{\overline{\lambda} \in \mathbb{C} : \lambda \in \sigma(T)\right\}.$$

Proof. If $\lambda \in \rho(T)$, then $T - \lambda I$ is invertible, consequently, $(T - \lambda I)^* = T^* - \overline{\lambda} I$ is invertible. Thus, $\overline{\lambda} \in \rho(T^*)$. Similarly, if $\overline{\lambda} \in \rho(T^*)$ then $\overline{\overline{\lambda}} = \lambda \in \rho(T)$. Therefore, $\lambda \in \rho(T) \Longleftrightarrow \overline{\lambda} \in \rho(T^*)$ which is equivalent to $\lambda \in \sigma(T) \Longleftrightarrow \overline{\lambda} \in \sigma(T^*)$. □

Example 4.3. *Let* $S : \ell^2 \longrightarrow \ell^2$ *be defined by* $S(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$. *Then: if* $\lambda \in C$, $|\lambda| < 1$ *so* λ *is an eigenvalue of* S^* *and* $\sigma(S) = \{\lambda \in \mathbb{C}; |\lambda \leq 1|\}$.

Solution. 1) Let $\lambda \in C$, $|\lambda| < 1$. So that λ is an eigenvalue of S^*, it suffices that exists $0 \neq x \in \ell^2$, such that $S^* x = \lambda x$, then

$$(x_2, x_3, x_4, \cdots) = (\lambda x_1, \lambda x_2, \lambda x_3, \cdots),$$

consequently,

$$(x_2, x_3, x_4, \cdots) = \left(\lambda x_1, \lambda^2 x_1, \lambda^3 x_1, \cdots\right),\ x_1 \neq 0.$$

So that $(\lambda x_1, \lambda^2 x_1, \lambda^3 x_1, \cdots) \in \ell^2$ it is necessary that

$$x_1^2 \sum_{n\geq 1} |\lambda^n|^2 = x_1^2 \sum_{n\geq 1} |\lambda|^{2n} < \infty,$$

which satisfies only for $|\lambda| < 1$. Thus, λ is an eigenvalue of S^* with eigenvector $x = (\lambda, \lambda^2, \lambda^3, \cdots)$.

2) We have $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma(S^*)$, but $\sigma(S^*)$ is closed, then

$$\overline{\{\lambda \in \mathbb{C} : |\lambda| < 1\}} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \subset \sigma(S^*).$$

From the above lemma we have, $\left\{\overline{\lambda} \in \mathbb{C} : |\lambda| \leq 1\right\} \subset \sigma(S)$ then $\left\{\overline{\lambda} \in \mathbb{C} : |\lambda| \leq 1\right\} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \subset \sigma(S)$.

On the other hand, since $\|S\| = 1$. If $|\lambda| > 1$, then $\lambda \notin \sigma(S)$ consequently, $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

Theorem 4.2. *Let H be a Hilbert space over C, and $T \in \mathscr{L}(H)$. then,*

1) *For any polynomial p, we have $\sigma(p(T)) = p(\sigma(T)) = \{p(\lambda); \lambda \in \sigma(T)\}$.*

2) *If T is invertible $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.*

Demonstration. *1) Let $q(z) = p(z) - p(\lambda)$, since $q(\lambda) = 0$, then $q(z) = (z - \lambda) r(z)$ and $q(T) = (T - \lambda I) r(T)$, where $r(z)$ is a polynomial.*

If $\lambda \in \sigma(T)$ then $(T - \lambda I)$ is not invertible, consequently, $q(T) = p(T) - p(\lambda) I = (T - \lambda I) r(T)$ is not invertible, therefore, $p(\lambda) \in \sigma(p(T))$.

Reciprocally, let $\lambda \in \sigma(p(T))$ and define the polynomial $q(z) = p(z) - \lambda$. The polynomial $p(z)$ can be written $q(z) = c(z - \mu_1)(z - \mu_2) \cdots (z - \mu_n)$ for some $c \neq 0$ and $\mu_1, \mu_2, \cdots, \mu_n \in C$. Since $\lambda \in \sigma(p(T))$ then $q(T) = p(T) - \lambda I$ is not invertible, accordingly, there exists $1 \leq i \leq n$ such that $T - \mu_i I$ is not invertible. Therefore, $\mu_i \in \sigma(T)$. On the other hand, $q(\mu_i) = p(\mu_i) - \lambda = 0$, which gives $\lambda = p(\mu_i) \in P(\sigma(T))$.

2) Since T is invertible, then $0 \notin \sigma(T)$. Thus, every $\lambda \in \sigma(T)$ can be written $\lambda = \mu^{-1}$, and we have

$$T^{-1} - \mu I = -\mu T^{-1}(T - \lambda I),$$
$$T - \lambda I = -\lambda T\left(T^{-1} - \mu I\right)$$

where $-\mu T^{-1}$ and λT are invertibles. Consequently, $T^{-1} - \mu I$ is not invertible if and only if $T - \lambda I$ is not invertible. Thus,

$$\mu = \lambda^{-1} \in \sigma\left(T^{-1}\right) \Longleftrightarrow \lambda \in \sigma(T)$$

and $\sigma(T^{-1}) = \{\lambda^{-1} \in \mathbb{C} : \lambda \in \sigma(T)\}$.

Corollary 4.1. *Let $U \in \mathscr{L}(H)$ be a unitary operator, then the spectrum of U is*

$$\sigma(U) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Proof. Since U is unitary, then $\|U\| = \|U^*\| = 1$ and $U^{-1} = U^*$. By the use of Theorem 4.2, we get

$$\sigma(U) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}, \; \sigma(U^*) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

On the other hand

$$\begin{aligned}\sigma(U) &= \left\{\lambda \in \mathbb{C} : \lambda^{-1} \in \sigma\left(U^{-1}\right) = \sigma(U^*)\right\} \\ &= \left\{\lambda \in \mathbb{C} : \left|\lambda^{-1}\right| \leq 1\right\} = \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}.\end{aligned}$$

Therefore, the result follows. □

Definition 4.3. *Let H be a Hilbert space, $T \in \mathscr{L}(H)$ and $\sigma(T)$ the spectrum of T.*

1) *The spectral radius of T, denoted by $r_\sigma(T)$, is the real number given by*

$$r_\sigma(T) := \sup\{|\lambda| \, ; \lambda \in \sigma(T)\}.$$

2) *The numerical range of T, denoted by $W(T)$, is defined by*

$$W(T) := \{\langle Tx, x\rangle \, ; \|x\| = 1\}.$$

Example 4.4. *If $U \in \mathscr{L}(H)$ is unitary, then, $r_\sigma(U) = 1$.*

Remark 4.3. *Clearly, we have $r_\sigma(T) \leq \|T\|$.*

Theorem 4.3. *Let $T \in \mathscr{L}(H)$ then,*

$$r_\sigma(T) = \lim_{n \longrightarrow \infty} \|T^n\|^{\frac{1}{n}} = \inf_{n \geq 1} \|T^n\|^{\frac{1}{n}}.$$

Demonstration. *Note by $r = \inf_{n \geq 1} \|T^n\|^{\frac{1}{n}}$, then, it is clear that $\lim_{n \longrightarrow \infty} \|T^n\|^{\frac{1}{n}} \geq r$. Let's prove that*

$$\lim_{n \longrightarrow \infty} \|T^n\|^{\frac{1}{n}} = r.$$

It suffices to prove that $\lim_{n \longrightarrow \infty} \|T^n\|^{\frac{1}{n}} \leq r$.

For all $\varepsilon > 0$, there exists $m \geq 1$ such that

$$\|T^m\|^{\frac{1}{m}} < r + \varepsilon.$$

Any $n \in N^*$ can be written in a unique way $n = mp + q$ avec $0 \leq q \leq m - 1$. Thus

$$\begin{aligned}\|T^n\|^{\frac{1}{n}} &= \left\|T^{mp+q}\right\|^{\frac{1}{n}} \leq \|T^{mp}\|^{\frac{1}{n}} \|T^q\|^{\frac{1}{n}} \\ &\leq \|T^m\|^{\frac{p}{n}} \|T\|^{\frac{q}{n}} \leq (r + \varepsilon)^{\frac{mp}{n}} \|T\|^{\frac{q}{n}}.\end{aligned}$$

When $n \longrightarrow \infty$, then $p \longrightarrow \infty$ and consequently $\frac{mp}{n} = \frac{mp}{mp+q} \longrightarrow 1$ and $\frac{q}{n} \longrightarrow 0$. Therefore, we get

$$\lim_{n \longrightarrow \infty} \|T^n\|^{\frac{1}{n}} \leq r + \varepsilon.$$

Since ε is arbitrarely one gets $\lim_{n\longrightarrow\infty} \|T^n\|^{\frac{1}{n}} \leq r$ *and consequently* $\lim_{n\longrightarrow\infty} \|T^n\|^{\frac{1}{n}} = r.$

On the other hand, since $\|T^n\| \leq \|T\|^n$ *we get* $\lim_{n\longrightarrow\infty} \|T^n\|^{\frac{1}{n}} \leq \|T\|.$

Let $\lambda \in C$, $|\lambda| > r$, *there exists* $\delta > 0$ *such that* $|\lambda| = r + \delta$. *Soit* ε *et* n *tels que* $\varepsilon < \delta$ *and* $\|T^n\|^{\frac{1}{n}} < r + \varepsilon$. *Therefore*

$$\left\|\frac{T}{\lambda}\right\|^n = \left(\frac{\|T\|}{|\lambda|}\right)^n \leq \left(\frac{r+\varepsilon}{r+\delta}\right)^n.$$

The serie of general term $\frac{T}{\lambda}$ *converges and we have*

$$-(T-\lambda I)\sum_{n\geq 0}\left(\frac{T}{\lambda}\right)^n = -\lim_{k\longrightarrow\infty}(T-\lambda I)\frac{1}{\lambda}\sum_{n=0}^{k}\left(\frac{T}{\lambda}\right)^n = I - \frac{1}{\lambda}\lim_{k\longrightarrow\infty}\left(\frac{T}{\lambda}\right)^k = I.$$

Consequently $T - \lambda I$ *is invertible and* $\lambda \in \rho(T)$. *This shows that* $r \geq r_\sigma(T)$, *which completes the proof.*

4.2 Spectrum of some operators

Lemma 4.4. *Let H be a complex Hilbert space and let $T \in \mathscr{L}(H)$ be a normal operator, then*

$$\sigma(T) \subset \overline{W(T)}.$$

Proof. Let $\lambda \in \sigma(T)$, since $T - \lambda I$ is normal and non invertible, we have

$$\forall\alpha > 0, \exists x \in \mathcal{H}; \|(T-\lambda I)x\| < \alpha\|x\|,$$

therefore, we can choose a sequence $\{x_n\}$ with $\|x_n\| = 1$ such that

$$\lim_{n\longrightarrow\infty}\|(T-\lambda I)x_n\| = 0,$$

consequently

$$\lim_{n\longrightarrow\infty}|\langle(T-\lambda I)x_n, x_n\rangle| \leq \lim_{n\longrightarrow\infty}\|(T-\lambda I)x_n\|\|x_n\| = 0,$$

$$\lim_{n\longrightarrow\infty}\langle(T)x_n, x_n\rangle - \lim_{n\longrightarrow\infty}\lambda\langle x_n, x_n\rangle = 0.$$

Thus,

$$\lim_{n\longrightarrow\infty}\langle(T)x_n, x_n\rangle = \lambda,$$

which shows that $\lambda \in \overline{W(T)}$ and the proof is completes. □

Theorem 4.4. *Let $T \in \mathscr{L}(H)$ be a self-adjoint operator, then*

1) $W(T) \subset \mathbb{R}$,

2) $\sigma(T) \subset \mathbb{R}$,

3) *at least* $\|T\|$ *or* $-\|T\| \in \sigma(T)$,

4) $r_\sigma(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \|T\|$.

5) *for any* $\lambda \in W(T)$ *we have* $\inf \sigma(T) \leq \lambda \leq \sup \sigma(T)$.

Demonstration. *1) Since* T *is self-adjoint, then* $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$, *therefore* $\langle Tx, x\rangle \in \mathbb{R}$.

2) T *is self-adjoint hence,* T *is normal, and* $\sigma(T) \subset \overline{W(T)} \subset \mathbb{R}$.

3) If $T = 0$, *then* $0 \in \sigma(T)$. *Suppose that* $T \neq 0$ *and* $\|T\| = 1$, *then,* T^2 *is normal and there exists a sequence* $\{x_n\}$ *of unit vectors satisfies* $\lim_{n\longrightarrow\infty} \|Tx_n\| = 1$. *On the other hand we have*

$$\begin{aligned}
\left\|\left(I - T^2\right)x_n\right\|^2 &= \left\langle \left(I - T^2\right)x_n, \left(I - T^2\right)x_n\right\rangle \\
&= \|x_n\|^2 + \left\|T^2x_n\right\|^2 - 2\left\langle T^2x_n, x_n\right\rangle \\
&\leq \|x_n\|^2 + (\|T\|\,\|Tx_n\|)^2 - 2\left\langle T^2x_n, x_n\right\rangle \\
&\leq 2 - 2\langle Tx_n, Tx_n\rangle
\end{aligned}$$

consequently $\lim_{n\longrightarrow\infty} \|(I - T^2)x_n\|^2 = 0$, *there is no* $\alpha > 0$ *satisfies* $\|(I - T^2)x\| \geq \alpha\|x\|$ *then* $I - T^2$ *is not invertible, therefore* $1 \in \sigma(T^2) = (\sigma(T))^2$, *which entails that* $1 \in \sigma(T)$ *ou* $-1 \in \sigma(T)$.

If $\|T\| \neq 1$, *we set* $S = \|T\|^{-1} T$ *then* $\|S\| = 1$ *and proceed as before.*

4) From 3) and the previous lemma we conclude that $\|T\| \leq r_\sigma(T) \leq \sup\{|\lambda| : \lambda \in W(T)\}$.

From the inequality of Cauchy Schwarz we get $|\langle Tx, x\rangle| \leq \|Tx\|\,\|x\| \leq \|T\|\,\|x\|^2$, *then* $\sup\{|\lambda| : \lambda \in W(T)\} \leq \|T\|$, *which proves 4).*

5) Let $\lambda \in W(T)$ *and* $y \in H$, *such that* $\|y\| = 1$ *and* $\lambda = \langle Ty, y\rangle$. *Let* $\alpha = \inf \sigma(T)$ *and* $\beta = \sup \sigma(T)$. *Then,* $\sigma(\beta I - T) = \beta - \sigma(T) \subset [0, \beta - \alpha]$ *consequently* $r_\sigma(\beta I - T) \leq \beta - \alpha$. *Suppose* $\lambda < \alpha$, *then*

$$\langle(\beta I - T)y, y\rangle = \beta - \lambda > \beta - \alpha.$$

But from 4) $\beta - \alpha = r_\sigma(\beta I - T) = \sup\{\langle(\beta I - T)x, x\rangle, \|x\| = 1\} \geq \langle(\beta I - T)y, y\rangle = \beta - \lambda$, *this is a contradiction.*

Suppose that $\lambda > \beta$, *we get* $\sigma(T - \alpha I) = \sigma(T) - \alpha \subset [0, \beta - \alpha]$ *and* $r_\sigma(T - \alpha I) \subset [0, \beta - \alpha]$. *But*

$$\langle(T - \alpha I)y, y\rangle = \langle Ty, y\rangle - \alpha\langle y, y\rangle = \lambda - \alpha \geq \beta - \alpha.$$

Contradiction, which completes the proof.

Corollary 4.2. *Let* A *be a self adjoint matrix with eigenvalues* $\{\lambda_1, \lambda_2, \cdots \lambda_n\}$ *then,* $\|A\| = \sup\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\}$.

4.2.1 Positive operator

Definition 4.4. *Let $T \in \mathscr{L}(H)$. We say that T is positive if T is self-adjoint and $\langle Tx, x\rangle \geq 0,\ \forall x \in H$.*

If T is positive, we write $T \geq 0$, It $T - S$ is positive we write $T \geq S$.

Example 4.5. $0, I$ *et* TT^* *are positive.*

Lemma 4.5. *If T is self adjoint, then T is positive, if and only if $\sigma(T) \subset [0, +\infty[$.*

Proof. Suppose that T is positive, then $\sigma(T) \subset \overline{W(T)} \subset [0, +\infty[$. If $\sigma(T) \subset [0, +\infty[$ one get $0 = \inf \sigma(T) \leq \langle Tx, x\rangle \leq \sup \sigma(T)$ and hence T is positive. □

4.2.2 Projections

Definition 4.5. *An operator $P \in \mathscr{L}(H)$ is said to be orthogonal projection if $P = P^* = P^2$.*

Example 4.6. $P : \mathbb{C}^3 \longrightarrow \mathbb{C}^3,\ P(x, y, z) = (x, y, 0)$. *Since $\mathbb{C}^3$ is finite dimensional, P is continuous.*

$$\langle P(x, y, z), (u, v, w)\rangle = x\overline{u} + y\overline{v} = \langle (x, y, z), P(u, v, w)\rangle,$$

then $P = P^*$. Moreover, clearly $P = P^2$.

Lemma 4.6. *an orthogonal projection is positive.*

Proof. $\langle Px, x\rangle = \langle P^2x, x\rangle = \langle Px, P^*x\rangle = \langle Px, Px\rangle = \|Px\|^2$. □

4.3 Spectrum of compact operator

In what follows, we let H be a complex Hilbert space and $T \in \mathscr{K}(\mathcal{H})$ a compact operator.

Lemma 4.7. *If H is infinite dimensional and $T \in \mathscr{K}(\mathcal{H})$, then, $0 \in \sigma(T)$.*

Proof. If $0 \notin \sigma(T)$ then, T is invertible, which is in contradiction with the corollary 3.1 of compact operators chapter. □

Lemma 4.8. *If H is not separable, then $0 \in \sigma_p(T) = VP(T)$ is an eigenvalue of T.*

Proof. Since T is not separable, then $\overline{\mathrm{Im}(T)} \neq \mathcal{H}$, consequently, $\ker(T) = \overline{\mathrm{Im}(T)}^{\perp} \neq \{0\}$. Therefore, there exists $0 \neq e \in \ker(T),\ Te = 0$. which shows that 0 is an eigenvalue of T. □

Lemma 4.9. *Let $\lambda \neq 0$, then,* $\ker(T - \lambda I)$ *is of finite dimensional.*

Proof. Note that since $T-\lambda I$ is bounded, then $\ker(T-\lambda I)$ is closed, therefore $\ker(T-\lambda I)$ is a Hilbert subspace of $\mathcal{H}$. Suppose that $\dim(\ker(T-\lambda I))=\infty$, then, $\ker(T-\lambda I)$ is an infinite Hilbert space. Consequencly, there exists an orthonormal sequence $\{e_n\}\subset\ker(T-\lambda I)$. Note that $e_n\in\ker(T-\lambda I)$ then $Te_n=\lambda e_n$, hence, for any $n\neq m$,

$$\|\lambda e_n-\lambda e_m\|^2=\langle\lambda e_n-\lambda e_m,\lambda e_n-\lambda e_m\rangle=2\lambda^2.$$

The sequence $\{\lambda e_n\}$ is not a Cauchy sequence, therefore is not convergent, and T is not a compact operator, which is not true. □

Theorem 4.5. *For any* $\lambda\neq 0$, $\operatorname{Im}(T-\lambda I)$ *is closed and*

$$\operatorname{Im}(T-\lambda I)=\left(\ker\left(T^*-\overline{\lambda}I\right)\right)^{\perp}.$$

Demonstration. *Let* $\{y_n\}$ *be a sequence from* $\operatorname{Im}(T-\lambda I)$ *that converges to* $y\in H$ *and let* $\{x_n\}$ *be the sequence given by* $y_n=(T-\lambda I)x_n$.
Since $\ker(T-\lambda I)$ *is closed, then* $H=\ker(T-\lambda I)\oplus(\ker(T-\lambda I))^{\perp}$, *then the orthogonal decomposition of* x_n *with respect to* $\ker(T-\lambda I)$ *is* $x_n=u_n+v_n$ *where* $u_n\in\ker(T-\lambda I)$ *and* $v_n\in(\ker(T-\lambda I))^{\perp}$.

Our aim is to prove that $\{v_n\}$ *is bounded.*
Suppose that $\{v_n\}$ *is not bounded, then, we can extract from* (v_n) *a subsequence, which we keep denoted by* $\{v_n\}$ *for simplicity, such that* $\|v_n\|\neq 0$ *and* $\lim_{n\longrightarrow\infty}\|v_n\|=\infty$. *Set* $w_n=v_n/\|v_n\|$, *then* $\{w_n\}\subset(\ker(T-\lambda I))^{\perp}$ *and* $\|w_n\|=1$, *the sequence* $\{w_n\}$ *is bounded and we have*

$$(T-\lambda I)w_n=(T-\lambda I)\frac{x_n}{\|v_n\|}=\frac{y_n}{\|v_n\|}.$$

Thus, $\lim_{n\longrightarrow\infty}(T-\lambda I)w_n=\lim_{n\longrightarrow\infty}\frac{y_n}{\|v_n\|}=0.$
Since T *is compact, we can extract a subsequence* $\{w_{n_k}\}$ *such that* $\{Tw_{n_k}\}$ *converges. We infer then that*

$$\lim_{n\longrightarrow\infty}w_{n_k}=\frac{1}{\lambda}\left(\lim_{n\longrightarrow\infty}(T-\lambda I)w_{n_k}-Tw_{n_k}\right)=\frac{1}{\lambda}\lim_{n\longrightarrow\infty}Tw_{n_k}$$

which shows that $\{w_{n_k}\}$ *converges to* $w=\lim_{n\longrightarrow\infty}w_{n_k}$ *with* $\|w\|=1$.
Moreover, we have

$$(T-\lambda I)w=\lim_{n\longrightarrow\infty}(T-\lambda I)w_{n_k}=0$$

then, $w\in\ker(T-\lambda I)$. *But* $\{w_{n_k}\}\subset(\ker(T-\lambda I))^{\perp}$ *and consequently*

$$\|w-w_{n_k}\|^2=\langle w-w_{n_k},w-w_{n_k}\rangle=2,$$

which is in contraduction with $\lim_{n\longrightarrow\infty}w_{n_k}=w$. *Consequently* $\{v_n\}$ *must be bounded.*

Recall that T *is compact, then, we can suppose that* $\{Tv_{n_k}\}$ *converges. Therefore*

$$\lim_{n\longrightarrow\infty}v_{n_k}=\lim_{n\longrightarrow\infty}\frac{1}{\lambda}\left(Tv_{n_k}-(T-\lambda I)v_{n_k}\right)=\lim_{n\longrightarrow\infty}\frac{1}{\lambda}\left(Tv_{n_k}-y_{n_k}\right)$$

and hence $\{v_{n_k}\}$ *converges to* v.
We have

$$y = \lim_{n \longrightarrow \infty} (T - \lambda I) v_{n_k} = (T - \lambda I) v$$

which shows that $y \in \operatorname{Im}(T - \lambda I)$ *and finally,* $\operatorname{Im}(T - \lambda I)$ *is closed.*

Corollary 4.3. *If* $\lambda \neq 0$, *then*

$$\operatorname{Im}(T - \lambda I) = \ker\left(T^* - \overline{\lambda} I\right)^{\perp}$$

and

$$\operatorname{Im}\left(T^* - \overline{\lambda} I\right) = \ker(T - \lambda I)^{\perp}.$$

Proof. It suffices to use the well known results : $\operatorname{Im}(A) = \ker(A^*)^{\perp}$ and $\operatorname{Im}(A^*) = \ker(A)^{\perp}$. □

Lemma 4.10. *Let* $T \in \mathscr{K}(\mathcal{H})$, *then,* $\ker(T - I) = \{0\} \Longleftrightarrow \operatorname{Im}(T - I) = H$.

Proof. Let's prove that $\ker(T - I) = \{0\} \Longrightarrow \operatorname{Im}(T - I) = \mathcal{H}$. Suppose that $\operatorname{Im}(T - I) \neq \mathcal{H}$.
Set $H_1 = \operatorname{Im}(T - I) \subsetneq \mathcal{H}$, then H_1 is closed and the restriction of T on H_1 is compact, therefore $H_2 = (T - I) H_1$ is closed and furthermore T is injective and $H_2 \subsetneq H_1$.
By continuing the construction as above, we construct a decreasing sequence of closed subspeces $H_1 \supset H_2 \cdots \supset H_n$.
From Riesz's Lemma, there exists a sequencee $\{x_n\}$ $x_n \in H_n$ such that $\|x_n\| = 1$ and

$$\|x_n - y\| > \frac{1}{2}, \ \forall y \in H_{n+1}.$$

For any $n > m$ we have $(Tx_n - x_n) - (Tx_m - x_m) + x_n \in H_m$, then

$$\|Tx_n - Tx_m\| = \|((Tx_n - x_n) - (Tx_m - x_m) + x_n) - x_m\| > \frac{1}{2},$$

which is absurd, since T is compact. Therefore $\operatorname{Im}(T - I) = \mathcal{H}$.
Next, let's prove that $\operatorname{Im}(T - I) = \mathcal{H} \Longrightarrow \ker(T - I) = \{0\}$. Suppose that $\operatorname{Im}(T - I) = \mathcal{H}$, then $\ker(T^* - I) = \operatorname{Im}(T - I)^{\perp} = \{0\}$. Since T^* is compact, we can apply the first part of the proof on T^*, we entail then that $\operatorname{Im}(T^* - I) = \mathcal{H}$ and consequently, $\ker(T - I) = \{0\}$. The proof is completed. □

Let's admit without proof the followinf Lemma.

Lemma 4.11. *Let* $\{\lambda_n\} \subset \sigma(T) / \{0\}$ *be a sequence of distinct elements such that* $\lambda_n \longrightarrow \lambda$. *Then,* $\lambda = 0$.

Theorem 4.6. *Let* H *be an infinite dimensional Hilbert space and let* $T \in \mathscr{K}(\mathcal{H})$, *then,*

1) *If* $\lambda \in \sigma(T) \backslash \{0\}$, *then* $\lambda \in \sigma_P(T) = VP(T)$.

2) The spectrum of T is either:

i) $\sigma(T) = \{0\}$, *or*

ii) $\sigma(T) \setminus \{0\}$ *is finite, or*

iii) $\sigma(T) \setminus \{0\}$ *is a convergent sequence to* 0.

Demonstration. *1) If $\lambda \in \sigma(T)$ is not an eigenvalue, then* $\ker(T - \lambda I) = 0$, *and from Lemma 4.10,* $\operatorname{Im}(T - \lambda I) = H$. *Consequently, $T - \lambda I$ is invertible, hence $\lambda \in \rho(T)$ and this contradicts the fact that $\lambda \in \sigma(T)$.*

2) For $n \geq 1$, let E_n be given by

$$E_n = \sigma(T) \cap \left\{\lambda \in \mathbb{C}; |\lambda| \geq \frac{1}{n}\right\}.$$

E_n is a compact subset of $\sigma(T)$. If E_n contains an infinity of distinct elements, we can extract from E_n a convergent sequence to an element $\lambda \neq 0$ which contadicts the result of Lemme 4.11. Therefore, E_n is either empty or finite. Since $\sigma(T) / \{0\} = \underset{n\geq 1}{\cup} E_n$ we can arrange the elements of $\sigma(T) / \{0\}$ in a decreasing sequence $\{|\lambda_n|\}$.
If $\sigma(T) / \{0\}$ is infinite, the sequence $\{|\lambda_n|\}$ converges to 0.

4.3.1 Spectrum of compact self–adjoint operator

Corollary 4.4. *If T is compact and $\sigma(T) = \{0\}$ then $T = 0$.*

Proposition 4.1. *Suppose that H is separable and let $T \in \mathscr{K}(\mathcal{H})$ be a compact self–adjoint operator. Then, H admits a Hilbertian basis of eigenvectors of T.*

Demonstration. *Let $\sigma(T) / \{0\} = \{\lambda_1, \lambda_2, \cdots, \lambda_n, \cdot\cdot\}$. Denote by $\lambda_0 = 0$ and set $E_n = \ker(T - \lambda_n I)$. We have $\dim E_0 \leq \infty$ and $\dim E_n < \infty$ for all $n \geq 1$.*

1) The eigen spaces E_n are orthogonal anddisjoints of each others. Indeed, if $u \in E_n$ and $v \in E_m$ with $n \neq m$, we have $Tu = \lambda_n u$ et $Tv = \lambda_m v$ then

$$\langle Tu, v\rangle = \lambda_n \langle u, v\rangle = \langle u, Tv\rangle = \lambda_m \langle u, v\rangle$$

therefore $\langle u, v\rangle = 0$.

2) Let F be the space generated by $(E_n)_{n\geq 0}$. Then, F is dense in H. Indeed, we have $T(F) \subset F$, further, if $u \in F^\perp$ and $v \in F$ we get

$$\langle Tu, v\rangle = \langle u, Tv\rangle = 0.$$

Thus, $T\left(F^\perp\right) \subset F^\perp$.

3) Let T_0 be the restriction of T on $F^\perp$, T_0 is compact and self–adjoint. Moreover, if $\lambda \in \sigma(T_0) / \{0\}$ then $\lambda \in VP(T_0)$ and there exists a sequence $0 \neq u \in F^\perp$ such that $T_0 u = \lambda u$. Consequently, $\lambda = \lambda_n$ for $n \geq 0$ and therefore $u \in E_n \cap F^\perp = \{0\}$ which is absurd. Thus, $\sigma(T_0) = \{0\}$ and $T_0 = 0$. Consequently, $F^\perp \subset \ker T \subset F$ and we get $F^\perp = \{0\}$ which proves that F is dense in H.

4) The Hilbertian basis of H is the union of Hilbertian bases of E_n.

Chapter 5

Unbounde operators in Banach spaces

5.1 Introduction

Let X be a Banach space. In general an operator $T : X \to X$ is not necessarly defined on the whole space X but only on a subspace $D(T)$ called the domain of T.

Definition 5.1. *Let $D(T)$ be a subspace of X. An unbounded operator $T : D(T) \subset X \to X$ is a linear map T from $D(T)$ into X. $D(T)$ is the domai of T.*
The operator T is said to be continuous (bounded), if there exists a positive constant C such that

$$\|Tx\| \leq \|x\|, \ \forall x \in D(T).$$

Definition 5.2. *The graph of the operator T is the set*

$$\mathcal{G}(T) = \{(x, y) \in X \times X : x \in D(T), y = Tx\}.$$

Definition 5.3. *The operator $T : D(T) \subset X \to X$ is said to be closed if: for any convergent sequence $(x_n) \subset D(T)$ with $x_n \to x$, and $Tx_n \to y$, we have $x \in D(T)$ and $y = Tx$.*

Proposition 5.1 (Closed graph theorem)**.** *Let $T : D(T) \subset X \to X$ be a linear operator, if $\mathcal{G}(T)$ is closed in $X \times X$ then T is closed.*

Proposition 5.2. *Let X be a Banach space, and $T : D(T) \subset X \to X$ be a bounded linear operator, then T is closed.*

Demonstration. *Let $(x_n) \subset D(T)$ with $x_n \to x$, and $Tx_n \to y$. Since T is bounded, then*

$$y = \lim_{n \to +\infty} Tx_n = T(\lim_{n \to +\infty} x_n) = Tx.$$

Consequently, $x \in D(T)$ and $y = Tx$.

Remark 5.1. *Usually, $D(T)$ is endowed with the norm*

$$\|x\|_{D(t)} = \|x\|_X + \|Tx\|_X,$$

called graph norm.
If $D(T)$ is equipped with the graph norm, then any linear operator is bounded,

$$\|Tx\|_X \leq \|x\|_X + \|Tx\|_X = \|x\|_{D(t)}.$$

In what follows we suppose that $D(T)$ is endowed with the graph norm and the operator T is closed.

Definition 5.4. *The resolvent set of the operator T is set*

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda I : D(T) \to X \text{ is bijective}\}.$$

For any $\lambda \in \rho(T)$, the inverse $(T - \lambda I)^{-1}$ is, by the closed graph theorem, a bounded operator on X and is called the resolvent of T at λ and noted $R(\lambda, T)$ or $R(\lambda)$ if no confusion is feared.

Proposition 5.3. *Let X be a Banach space, then*
T is closed if and only if $D(T)$ is a Banach subspace of $X \times X$.

Lemma 5.1. *For $\lambda \in \rho(T)$ we have*

$$\lambda R(\lambda, T) = TR(\lambda, T) - I.$$

Demonstration.

$$\begin{aligned} T(T - \lambda I)^{-1} &= (T - \lambda I + \lambda I)(T - \lambda I)^{-1} \\ &= (T - \lambda I)(T - \lambda I)^{-1} + \lambda I (T - \lambda I)^{-1} \\ &= I + \lambda R(\lambda, T). \end{aligned}$$

As a result we deduce that $R(\lambda, T) T = TR(\lambda, T)$, because

$$\begin{aligned} R(\lambda, T) T &= R(\lambda, T)(T - \lambda I + \lambda I) \\ &= I + R(\lambda, T)(\lambda I) = I + \lambda R(\lambda, T). \end{aligned}$$

Lemma 5.2. *Let $\lambda, \mu \in \rho(T)$ then $R(\lambda, T)$ and $R(\mu, T)$ commute and*

$$R(\lambda, T) - R(\mu, T) = (\lambda - \mu) R(\lambda, T) R(\mu, T).$$

Demonstration.

$$\begin{aligned} (\lambda - \mu) R(\mu, T) R(\lambda, T) &= R(\mu, T)(\lambda I - \mu I) R(\lambda, T) \\ &= R(\mu, T)[(T - \mu I) - (T - \lambda I)] R(\lambda, T) \\ &= [I - R(\mu, T)(T - \lambda I)] R(\lambda, T) \\ &= R(\lambda, T) - R(\mu, T). \end{aligned}$$

Thus

$$\begin{aligned} R(\mu,T)\,R(\lambda,T) &= \frac{R(\lambda,T)-R(\mu,T)}{(\lambda-\mu)} \\ &= \frac{R(\mu,T)-R(\lambda,T)}{(\mu-\lambda)} \\ &= R(\lambda,T)\,R(\mu,T)\,. \end{aligned}$$

Definition 5.5. *As for the case of bounded operators, the spectrum of the operator T is the set*

$$\sigma(T) := \mathbb{C} - \rho(T) = \{\lambda \in \mathbb{C}; (T-\lambda I) \text{ is not invertible}\}\,.$$

The set $\sigma(T)$ is divided on three parts:

1) Punctual spectrum

$$\sigma_P(T) = \{\lambda \in \mathbb{C}; (T-\lambda I) \text{ is not injective}\}\,.$$

2) Continuous spectrum

$$\sigma_C(T) = \left\{ \begin{array}{c} \lambda \in \mathbb{C}; (T-\lambda I) \text{ is injective,} \\ Im(T-\lambda I) \neq E \text{ and is dense in } X \end{array} \right\}.$$

3) Residual spectrum

$$\sigma_R(T) = \left\{ \begin{array}{c} \lambda \in \mathbb{C}; (T-\lambda I) \text{ is injective,} \\ Im(T-\lambda I) \neq E \text{ and is not dense in } X \end{array} \right\}.$$

Example 5.1. *On $X = C\,[0,1]$ define T, S by $Tf = Sf$ with domain*

$$D(T) = C^1\,[0,1] \ \text{ and } D(S) = \left\{ f \in C^1\,[0,1]\,; f(1) = 0 \right\}.$$

Then $\sigma(T) = \mathbb{C}$, because for all $\lambda \in \mathbb{C}$, there exists $f(x) = e^{\lambda x}$ such that

$$(T-\lambda I)\,f(x) = 0.$$

$$\sigma(S) = \Phi$$

because, for all $f \in C\,[0,1]$,

$$R(\lambda,S)\,f(x) = -\int_x^1 e^{\lambda(x-y)} f(y)\,dy.$$

$$-S\int_x^1 e^{\lambda(x-y)} f(y)\,dy = -\lambda \int_x^1 e^{\lambda(x-y)} f(y)\,dy + f(x)$$

$$(S-\lambda I)\int_0^1 e^{\lambda(x-y)} f(y)\,dy = f(x)\,.$$

University of El Oued

Faculty of Exact sciences 2021/2022
Department of Mathematics Master 1 Maths
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Finish Exam on Spectrum Theory course

Exercise 1. (12 pts)
Let H be a complex Hilbert space and let $A \in \mathscr{L}(H)$ be a linear bounded operator. Suppose that there exist two self–adjoint operators S, T such that $A = S + iT$.

1) Determine S and T in terms of A and A^*.

2) Prove that
$$\mathcal{A}\mathcal{A}^* - \mathcal{A}^*\mathcal{A} = 2i(ST - TS).$$

3) Show that A is normal if and only if $ST = TS$.

3) Suppose that A is normal.

   i) Compute AA^* in term of $S^2 + T^2$ and prove that :
   A is invertible if and only if $S^2 + T^2$ is invertible.

   ii) Deduce that in this case (A normal) we have
$$\mathcal{A}^{-1} = \mathcal{A}^*(S^2 + T^2)^{-1}.$$

Exercise 2. (8 pts)

Let H be the Hilbert space $H = L^2([0, 1])$ and define the operator $T : H \longrightarrow H$ by
$$\mathcal{T}f(x) := \int_0^x (x - t)f(t)dt.$$

i) Compute the integral
$$I = \int_0^x |x - t|^2 dt.$$

ii) Prove that T is continuous and $\|T\| \leq \frac{1}{\sqrt{3}}$. (use Cauchy Shwarz).

iii) Let $g \in L^2([0, 1])$ be given. Prove that the equation
$$f(x) = g(x) + \int_0^x (x - t)f(t)dt,$$
has a unique solution, and express the solution as a function on T and g.

iv) Deduce that $1 \in \rho(T)$, the resolent set of T.

University of El Oued

Faculty of Exact sciences 2021/2022
Department of Mathematics Master 1 Maths
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Replay Exam on Spectrum Theory course

Exercise 1. (8 pts)
Let H be a complex Hilbert space and let $T \in \mathscr{L}(H)$ be a linear bounded operator.

a) Prove that $x \in H$, $x \neq 0$ is an eigenvector of T if and only if $|\langle Tx, x\rangle| = \|Tx\|\|x\|$.

b) Deduce that

$$\begin{pmatrix} \mathcal{T} \text{ has an eignenvalue } \lambda, \\ \text{with } |\lambda| = \|\mathcal{T}\|, \end{pmatrix} \Longleftrightarrow \exists x \neq 0; \|x\| = 1 \text{ and } |\langle \mathcal{T}x, x\rangle| = \|\mathcal{T}\|$$

Note: Recall that $(|\langle y, x\rangle| = \|y\|\|x\|) \Longleftrightarrow (\exists \lambda \in \mathbb{C};\ y = \lambda x)$.

Exercise 2. (12 pts)
Let $A : \ell^2(R) \longrightarrow \ell^2(R)$ be the operator defined by

$$\mathcal{A}\,(x_1, x_2, x_3, x_4 \cdots ,) = (0, 4x_1, x_2, 4x_3, x_4, \cdots)$$

1) Prove that A is bounded and deduce that

$$\|\mathcal{A}(x_n)\|_{\ell^2} \leq 4\|(x_n)\|_{\ell^2}.$$

2) Calculate $\|Ax_0\|$ for $x_0 = (1, 0, 0, \cdots)$ and prove that $\|A\| = 4$.
3) Find A^2 and calculate $\|A^2\|$. Then compare $\|A^2\|$ and $\|A\|^2$.
4) Determine A^* the adjoint of A.
5) Is the operator A normal.

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