Pascal Triangle or El-Kharji Triangle
History, Properties and Generalizations

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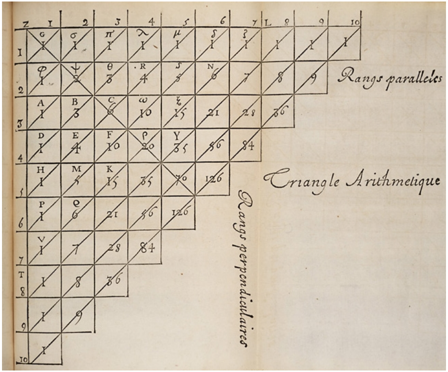
USTHB-CERIST

RECITS Laboratory, CATI Team

El Oued University, April 26th, 2022

European version

In the 17th century ***Blaise Pascal*** (1623 - 1662) published the BCT in his book *"Traité du triangle arithmétique"*[1].



[1] Edwards, A. W. F., The arithmetical triangle, Oxford University Press, 2013

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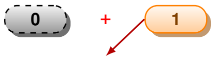

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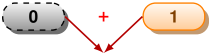

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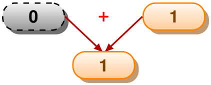

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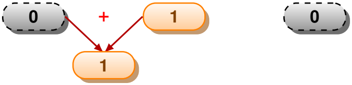

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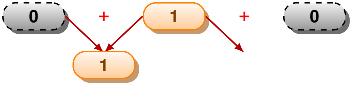

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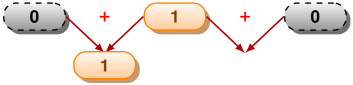

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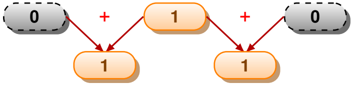

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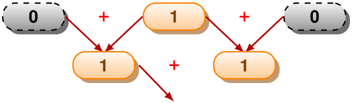

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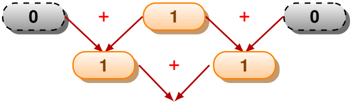

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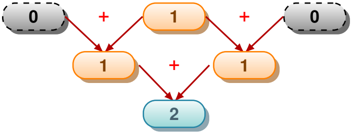

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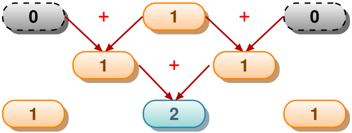

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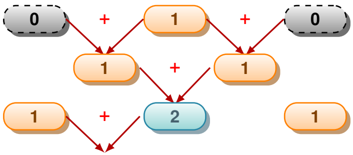

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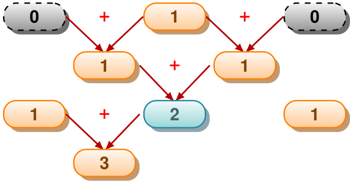

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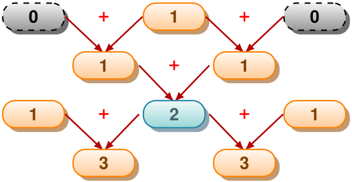

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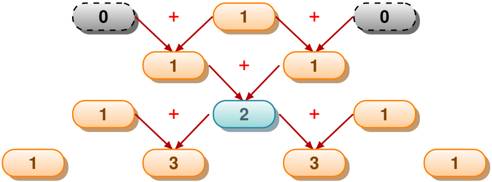

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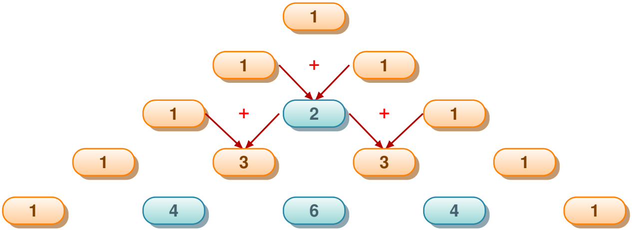

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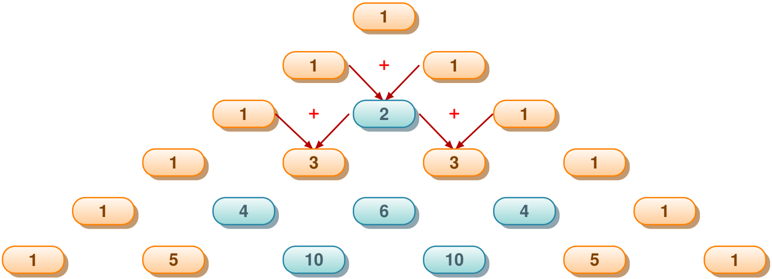

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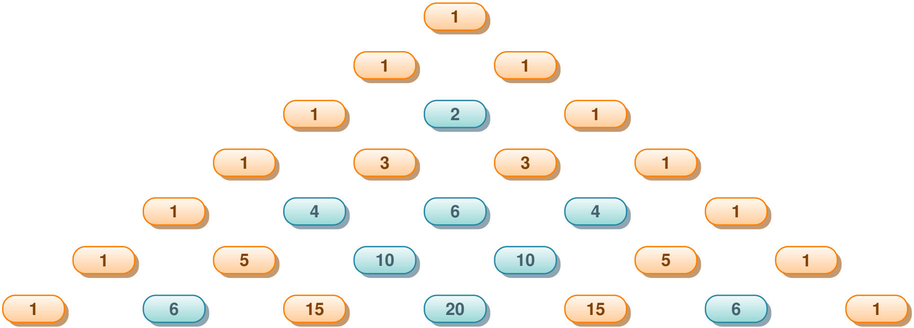

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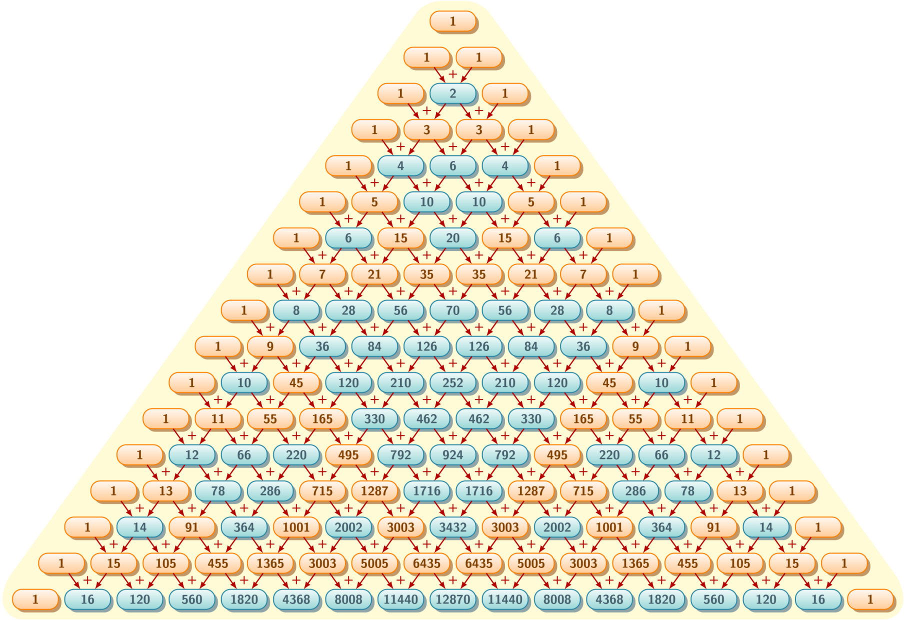

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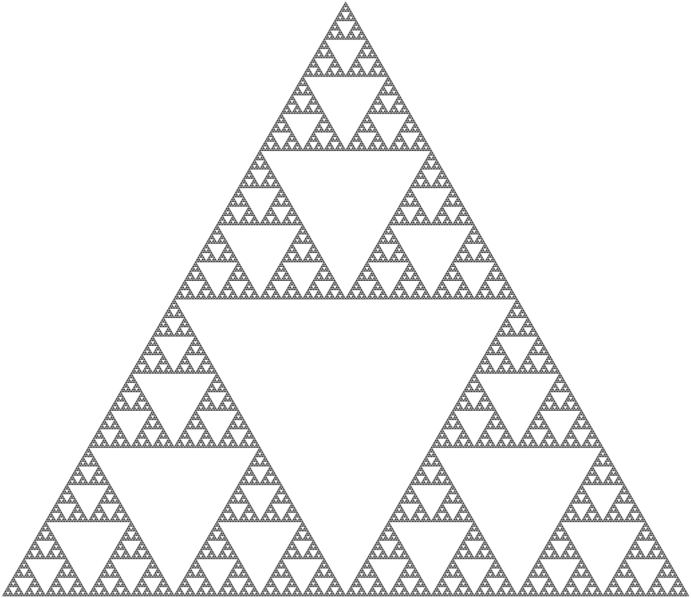
[2] See (https://texample.net/tikz/examples/pascals-triangle-and-sierpinski-triangle/)

Sierpinski triangle



Indian version

The earliest mention of binomial coefficients triangle (BCT), were known to ***Pingala*** (2nd century B.C).

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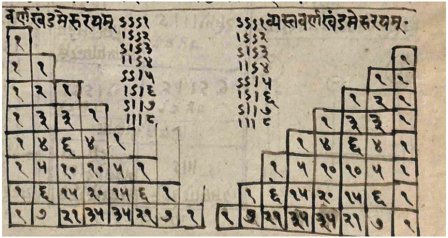

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Muslims version

"Al-Khalil Al-Farahidi" (718-786) used the binomial coefficients in his book *Kitab Al-'Ayn*.

[4] Evelyn Kennedy, The Mathematics Teacher, National Council of Teachers of Mathematics, OMAR KHAYYAM, 1966

[5] Rashed, Roshdi, Encyclopedia of the History of Arabic Science, 1996, Routledge

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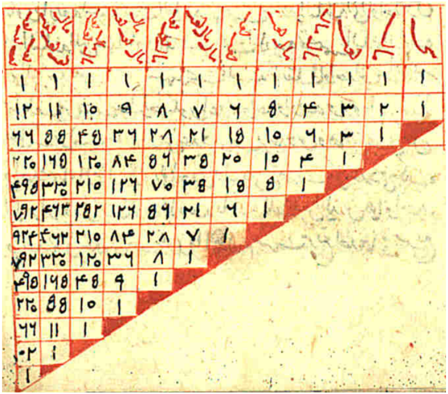

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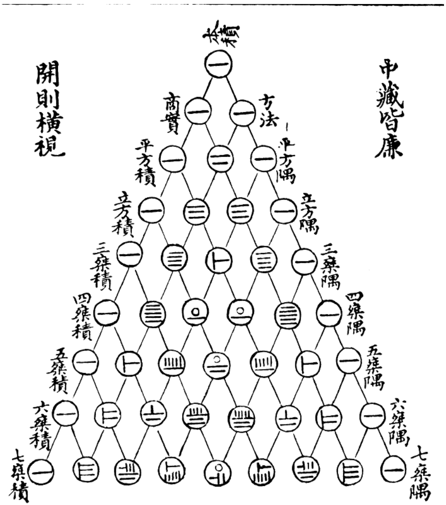
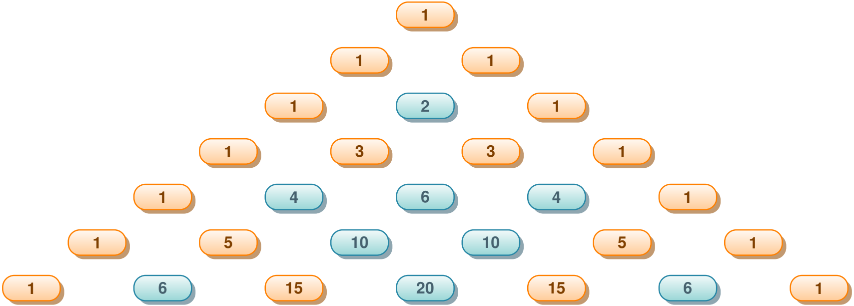
Chinese version

Another discovery of the BCT was made by the Chinese in the 11^{th} century and preserved through the work of the Chinese mathematician ***Yang Hui*** (1238-1298).



[5] Eric W. Weisstein,CRC Concise Encyclopedia of Mathematics, Chapman & Hall/CRC, 2002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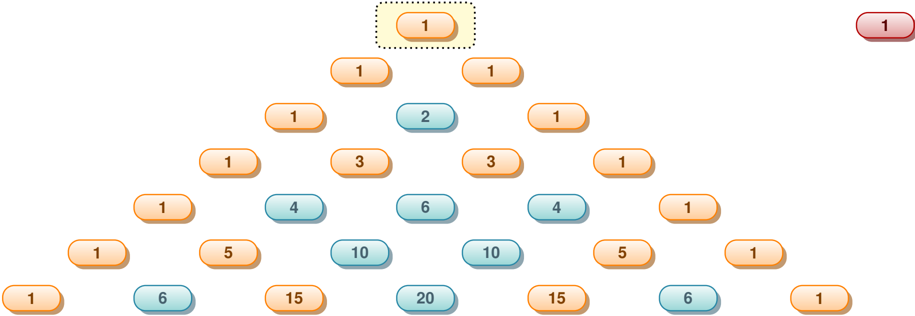
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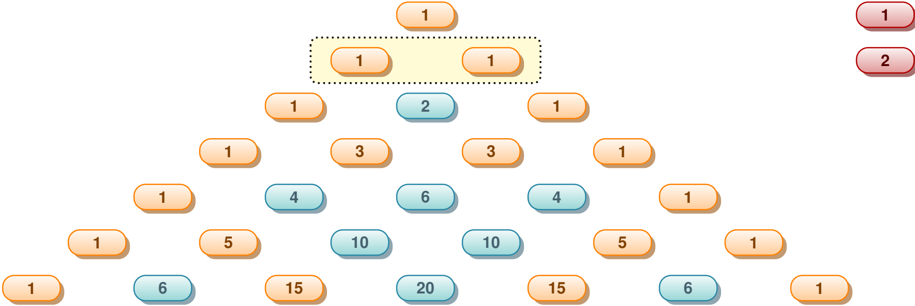

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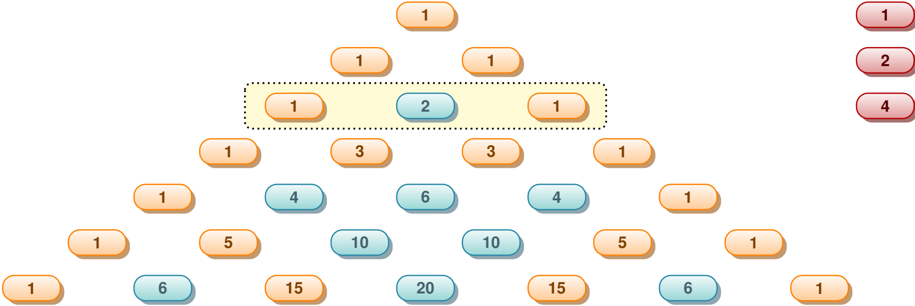
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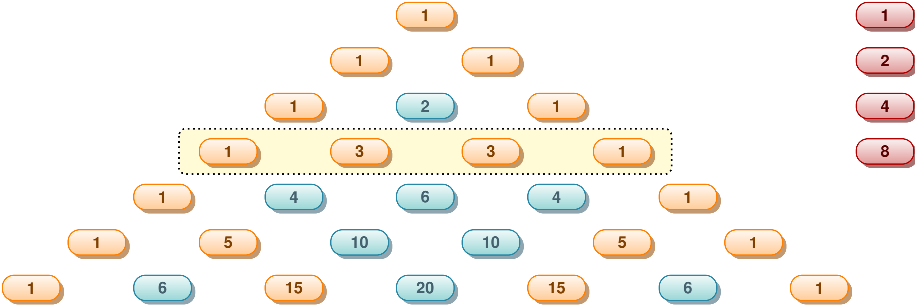
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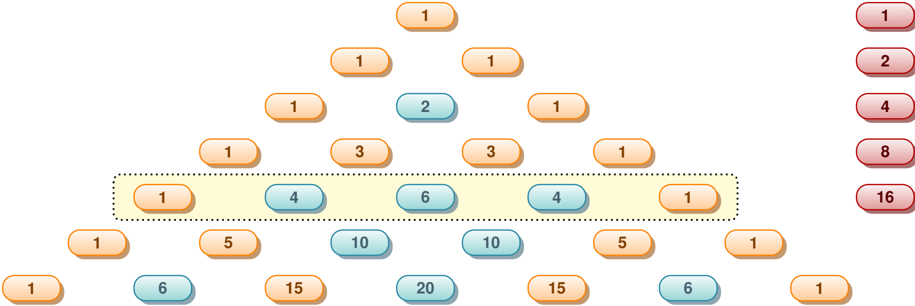
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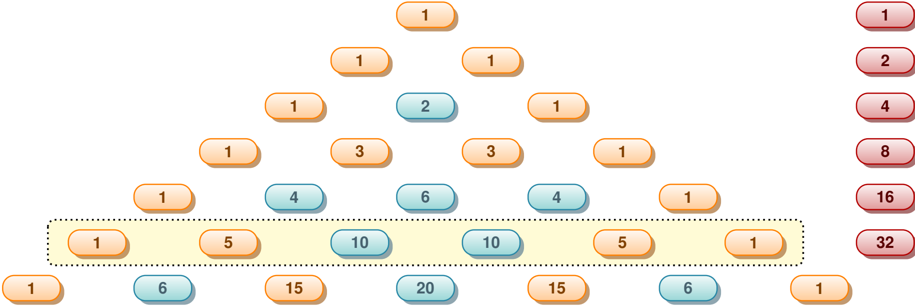

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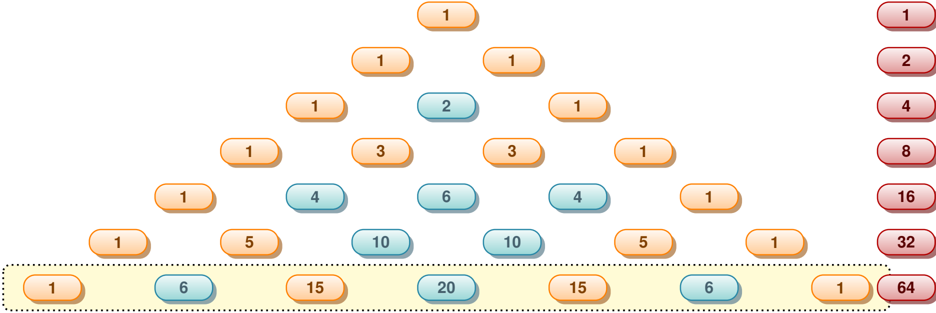

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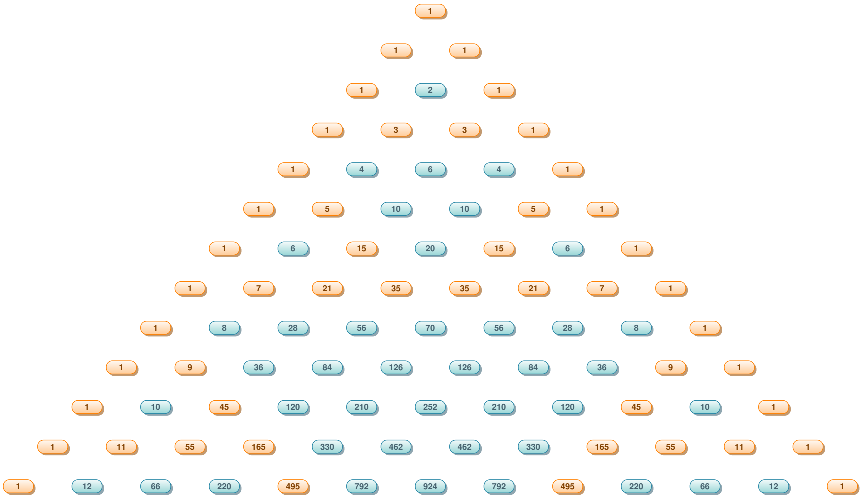
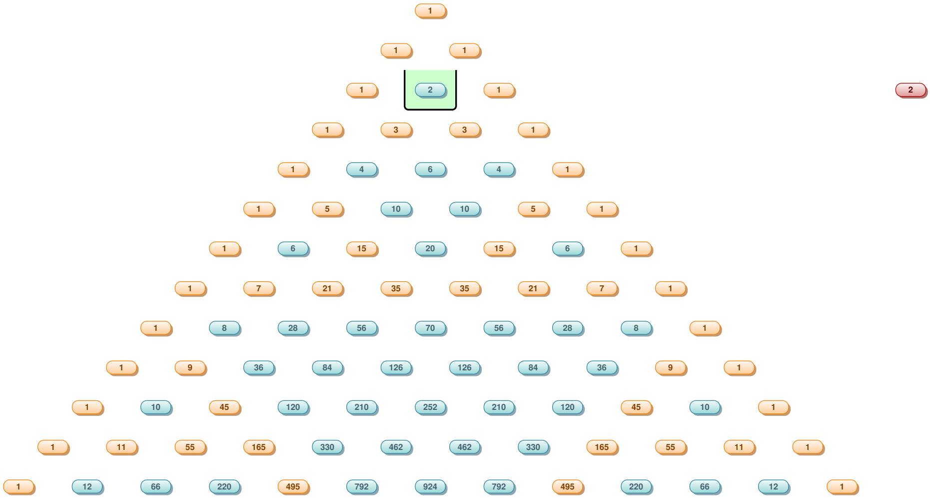


Ram’s theorem

$$\underset{0<k<n}{gcd}\binom{n}{k}\begin{cases} p, & for\ n = p^r,\ r > 0 \\ 1, & elsewhere. \end{cases}$$



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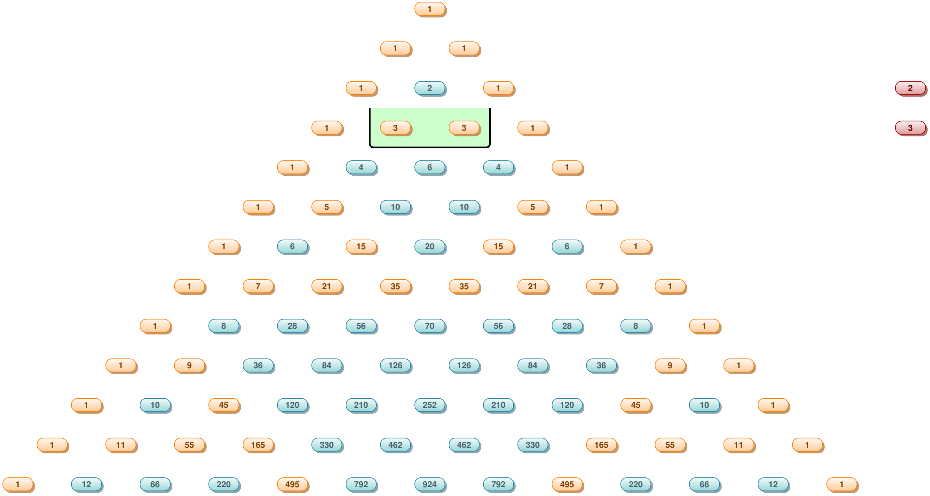

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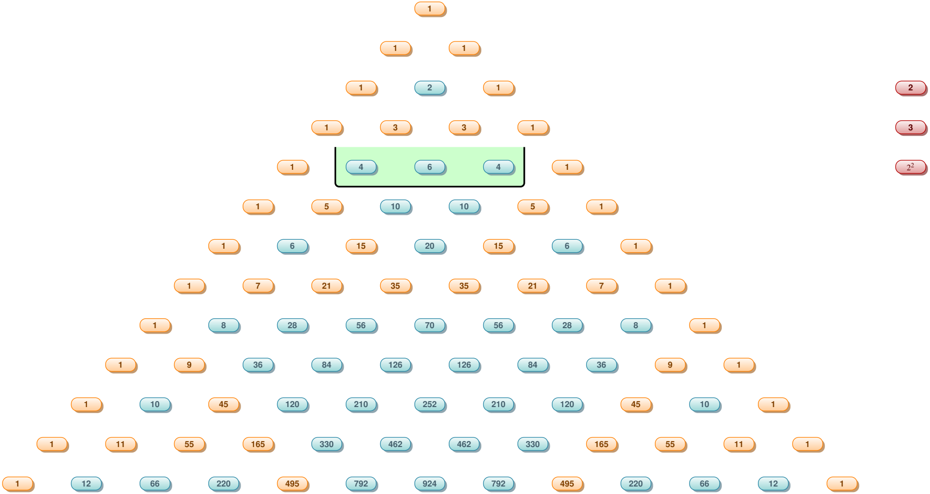

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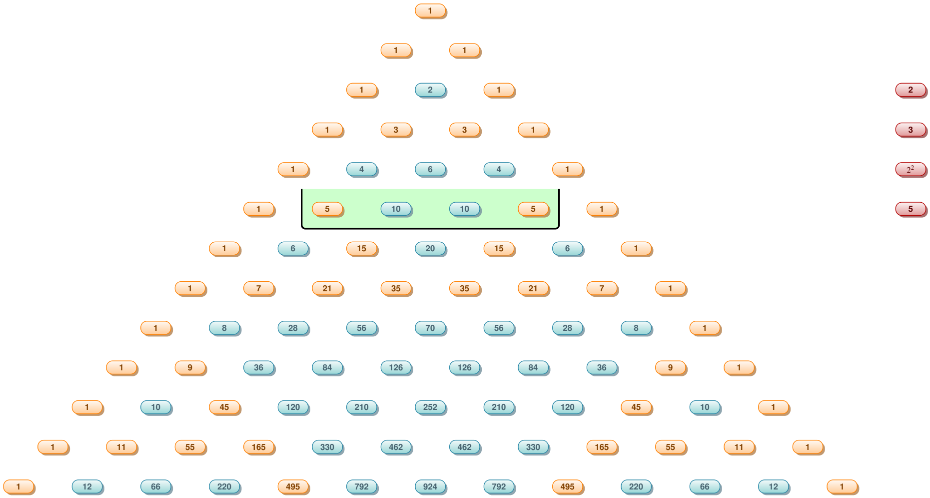

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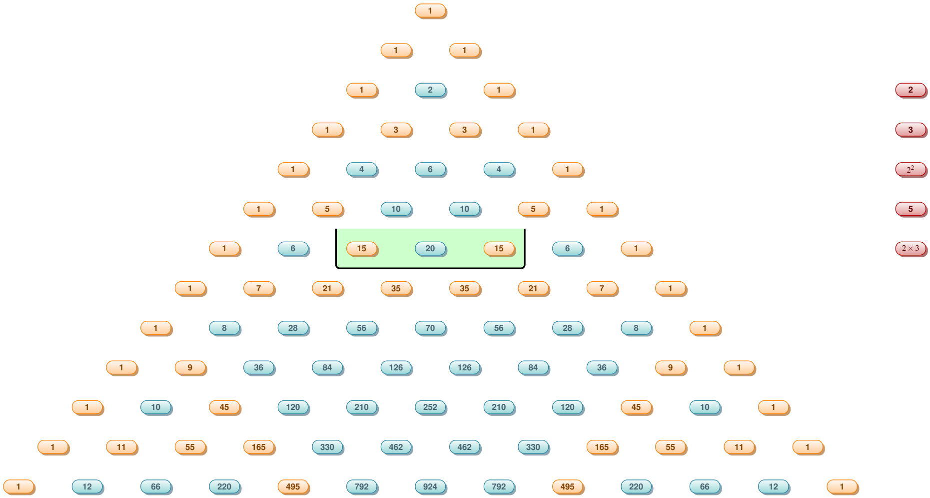

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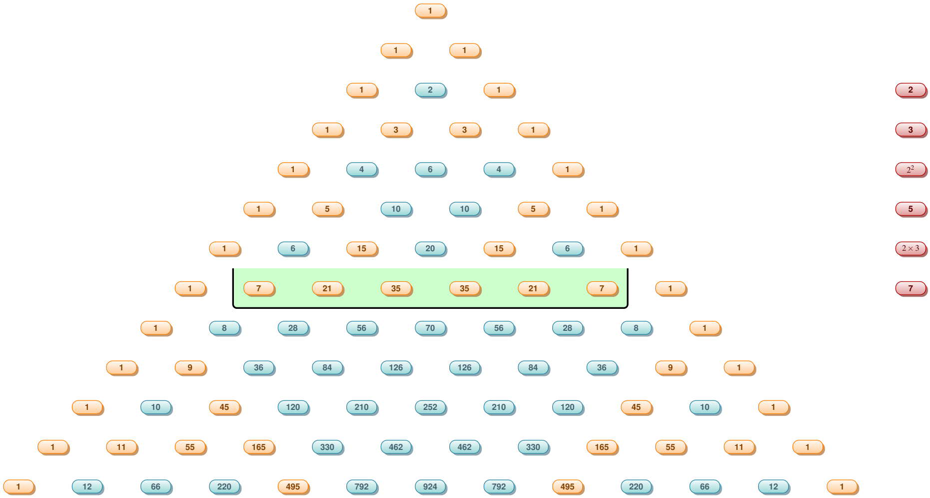

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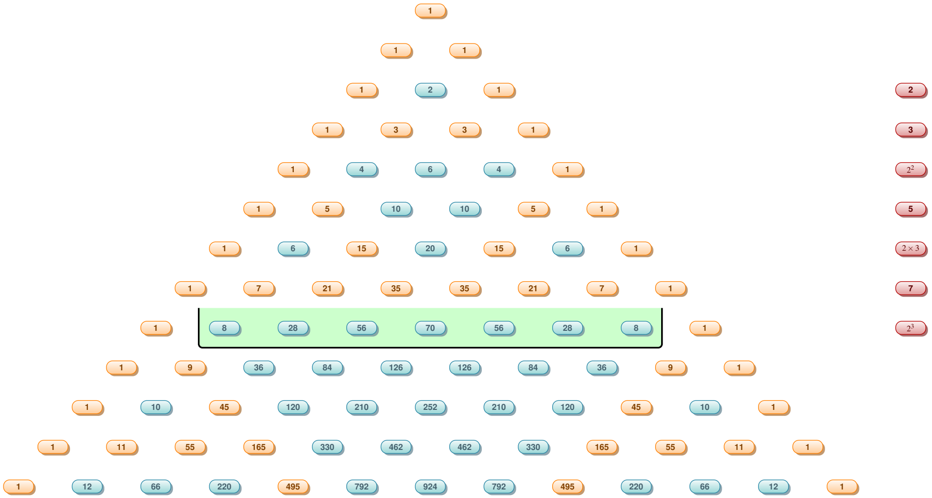

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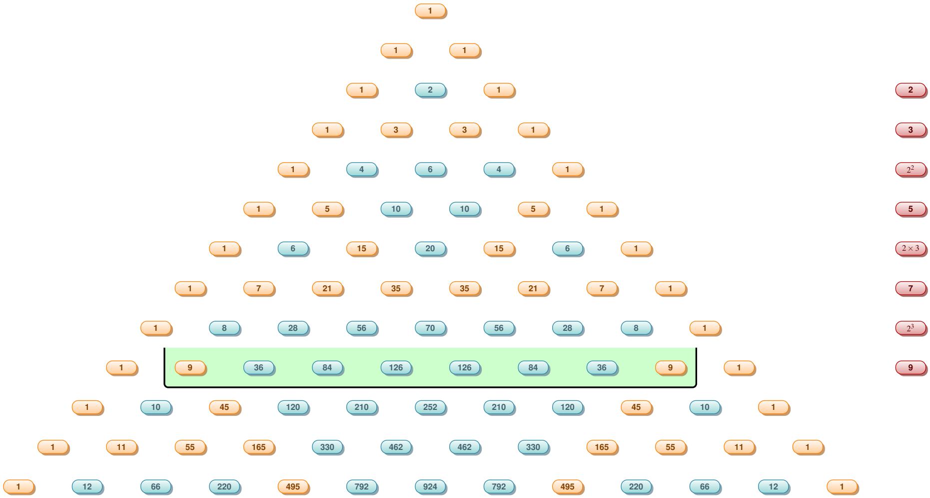

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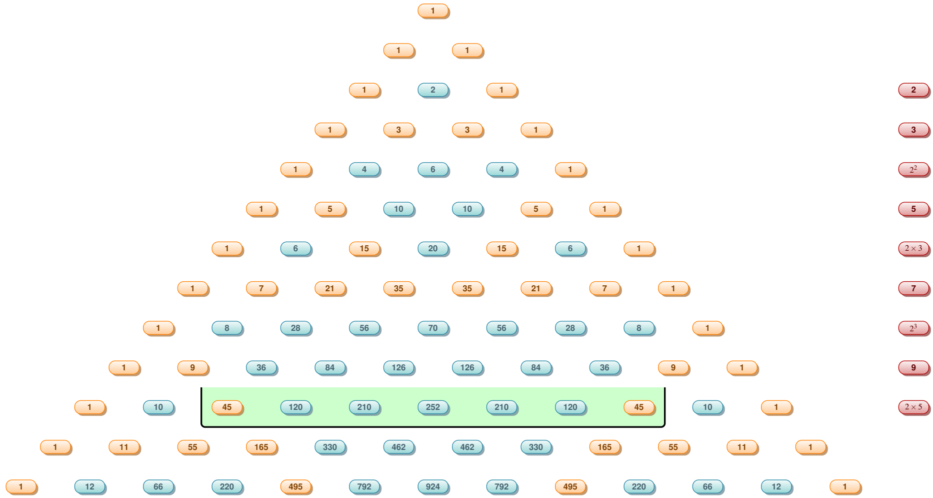

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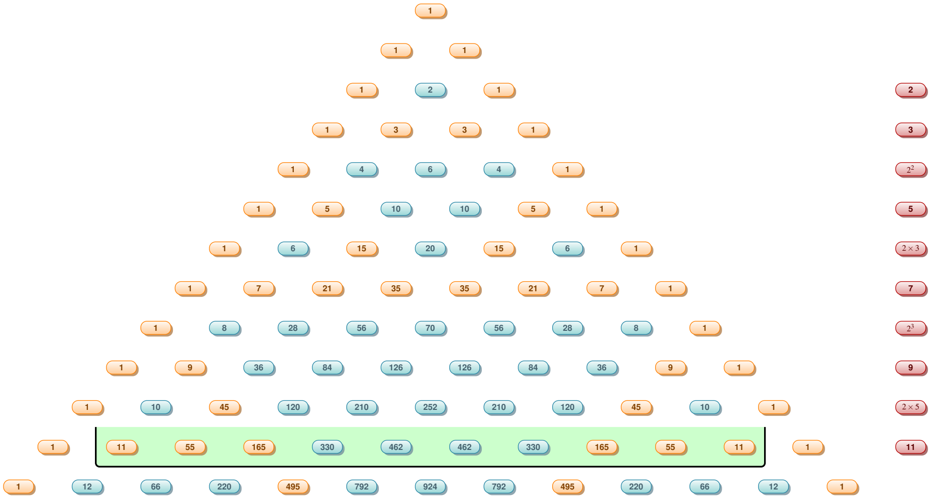

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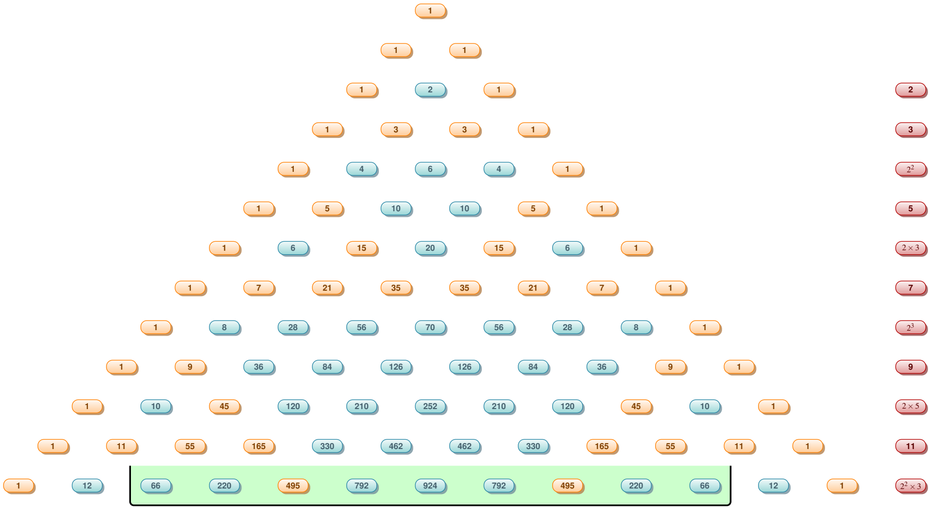

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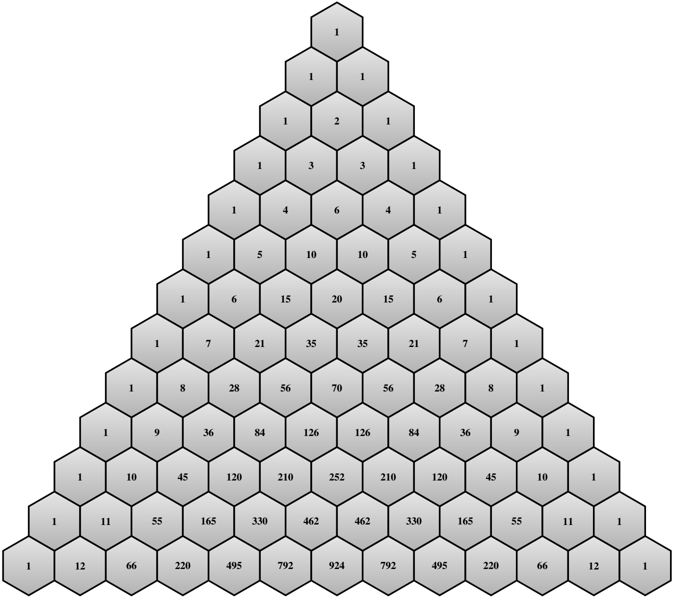
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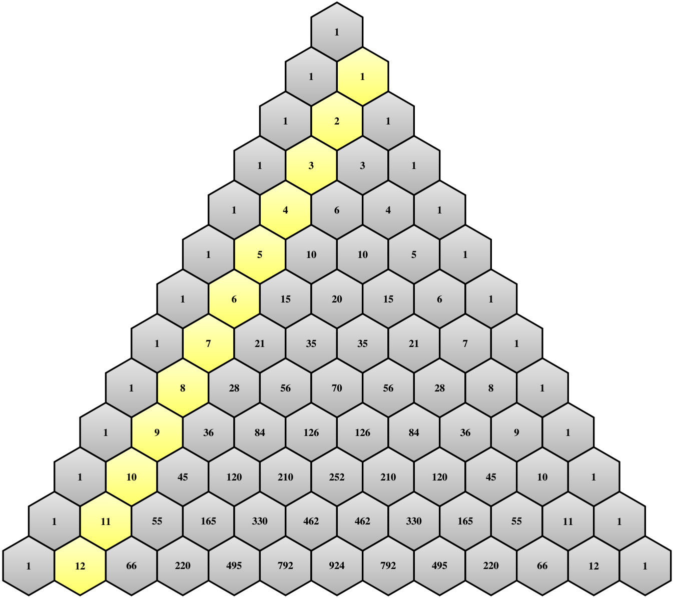
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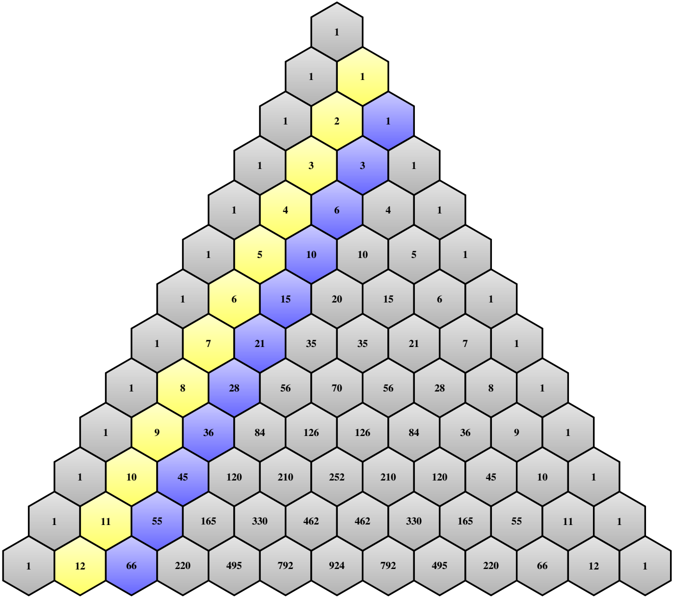
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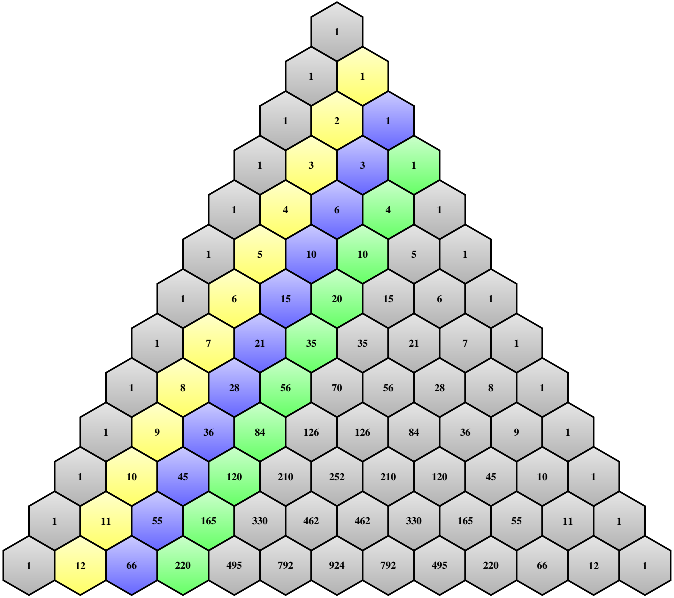
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1 6 15 20 15 6 1
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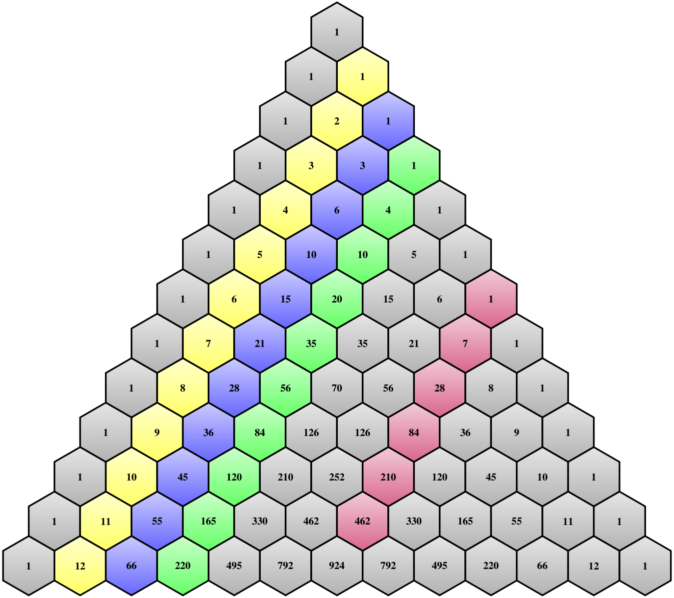
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1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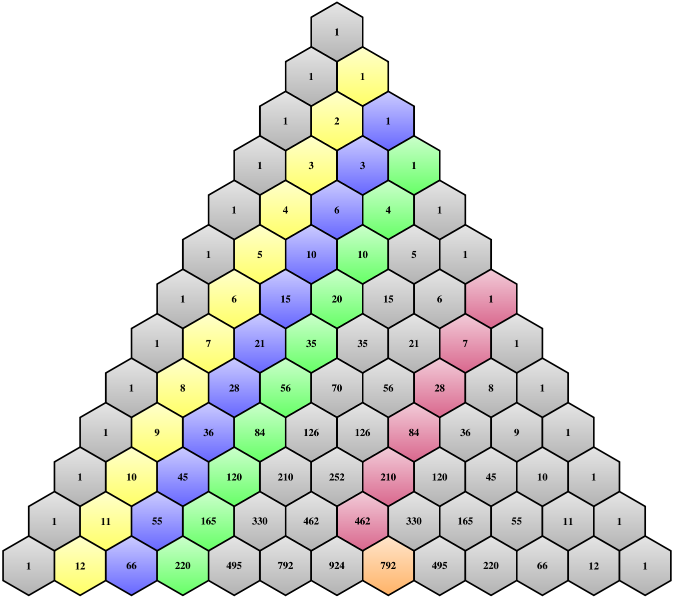
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Some binomial coefficients properties

The binomial coefficients appear in the expansion of

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the double generating function

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^k y^n = \frac{1}{1-y-xy}$$

Fibonacci sequence

1						
1	**1**					
1	**2**	**1**				
1	**3**	**3**	**1**			
1	**4**	**6**	**4**	**1**		
1	**5**	**10**	**10**	**5**	**1**	
1	**6**	**15**	**20**	**15**	**6**	**1**

$$F_n = \sum_{k=0}^{n} \binom{n-k}{k}$$

Fibonacci sequence

1

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

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Fibonacci sequence

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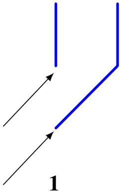

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1 2 1

1 3 3 1

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1 5 10 10 5 1

1 6 15 20 15 6 1

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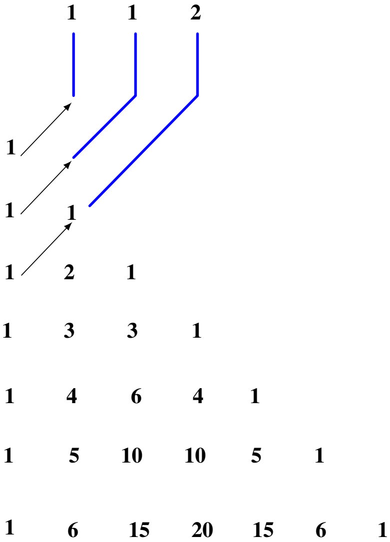
Fibonacci sequence



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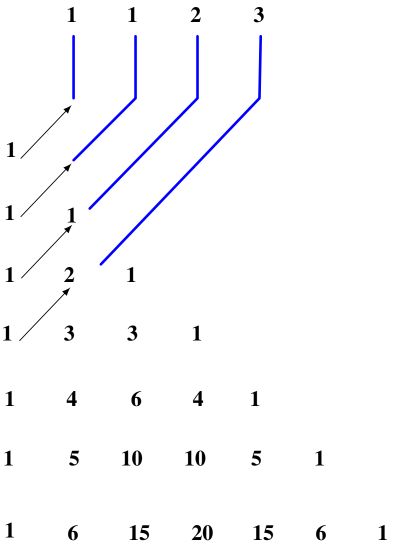
Fibonacci sequence



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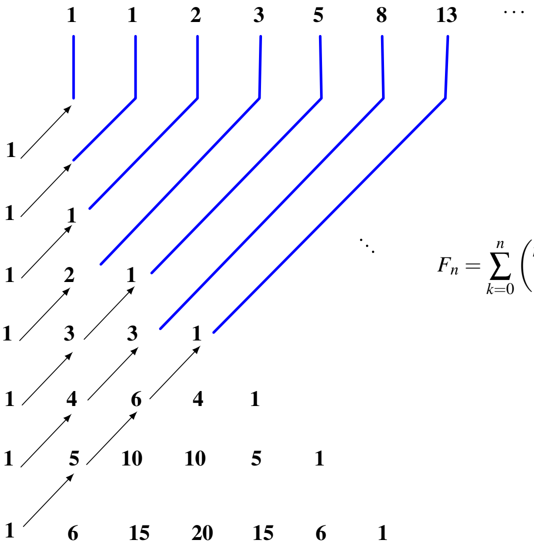
Fibonacci sequence



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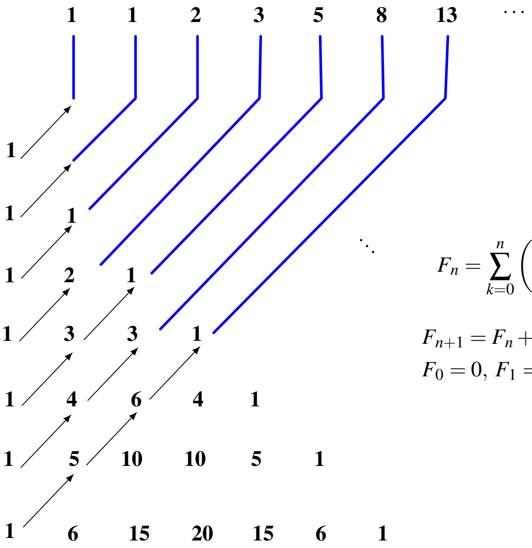

$$F_n = \sum_{k=0}^{n} \binom{n-k}{k}$$

$$F_{n+1} = F_n + F_{n-1}, \quad \text{for} \quad n \geq 2,$$
$$F_0 = 0,\ F_1 = 1.$$

Directions in Pascal triangle

In 1963 Raab [6] generalized it to diagonals of direction $(1, q)$ in the generalized Pascal triangle,

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$$U_n = \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \binom{n-qk}{k} x^{n-(q+1)k} y^k,$$

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Directions in Pascal triangle

In 2014[7] the recurrence relation for any given direction in generalized Pascal triangle was established,

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Directions in Pascal triangle

In 2014[7] the recurrence relation for any given direction in generalized Pascal triangle was established,

$$T_n^{(p,q,r)} = \sum_{k=0}^{\lfloor (n-p)/(q+r) \rfloor} \binom{n-qk}{p+rk} x^{n-p-(q+r)k} y^{p+rk}.$$

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satisfy the linear recurrence

$$T_n - x\binom{r}{1} T_{n-1} + x^2 \binom{r}{2} T_{n-2} + \cdots + (-1)^r x^r \binom{r}{r} T_{n-r} = y^r T_{n-r-q}.$$

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

1							
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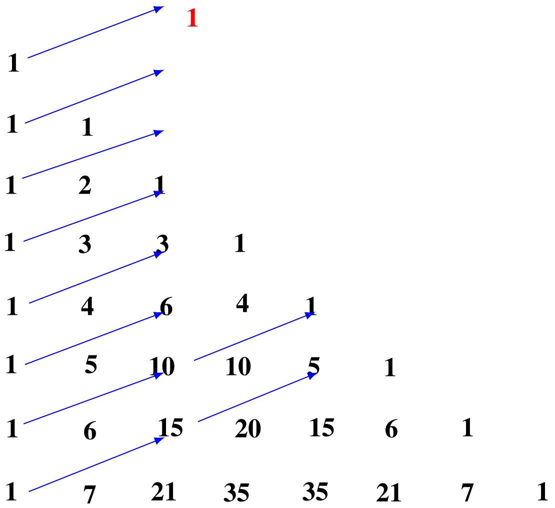
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1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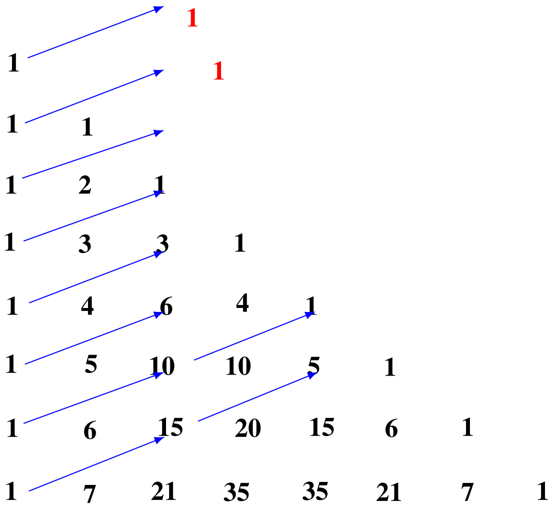
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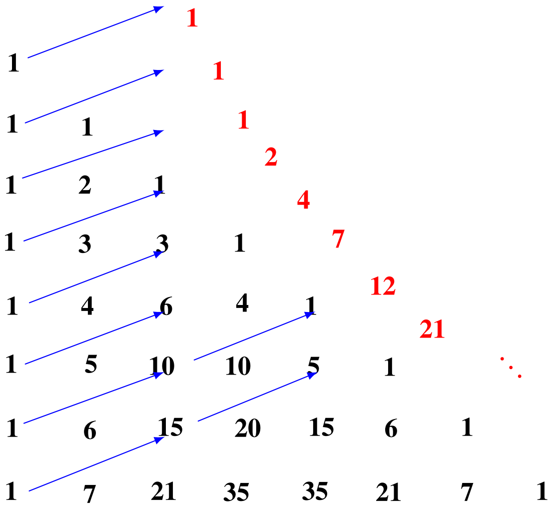
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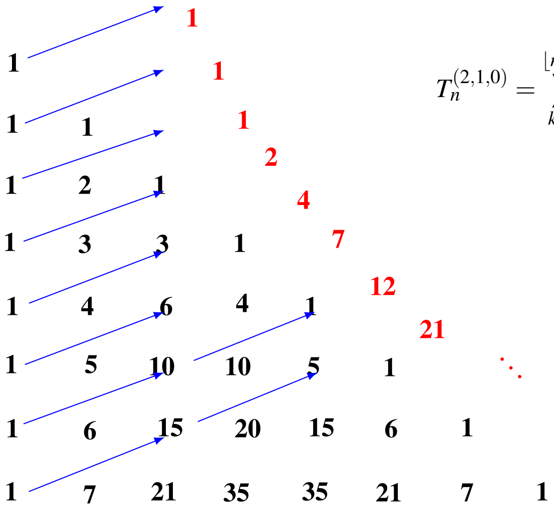
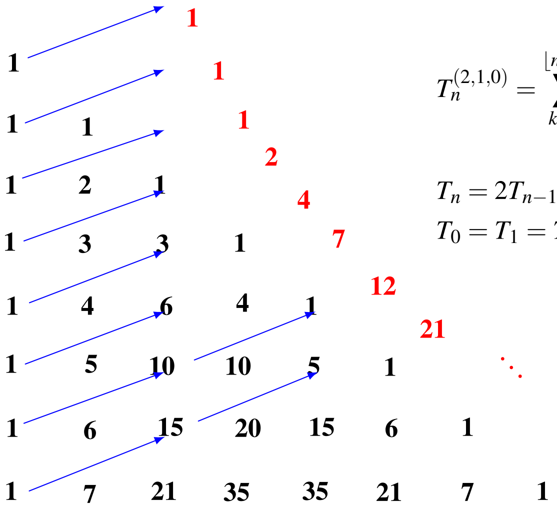


$$T_n^{(2,1,0)} = \sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n-k}{2k}$$



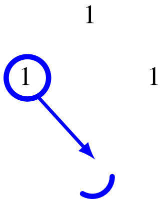
$$T_n^{(2,1,0)} = \sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n-k}{2k}$$

$$T_n = 2T_{n-1} - T_{n-2} + T_{n-3}$$
$$T_0 = T_1 = T_2 = 1$$

Delannoy triangle

1

Delannoy triangle



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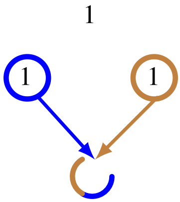

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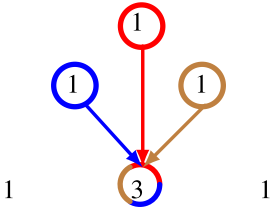

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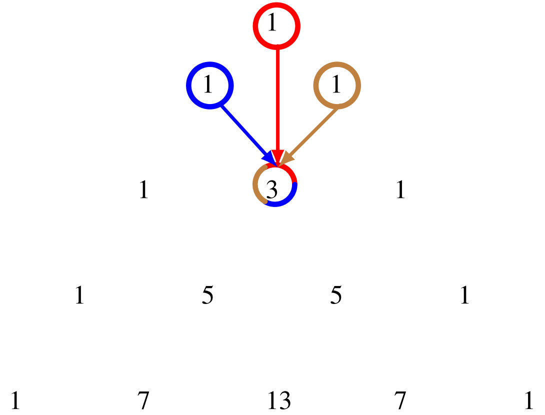

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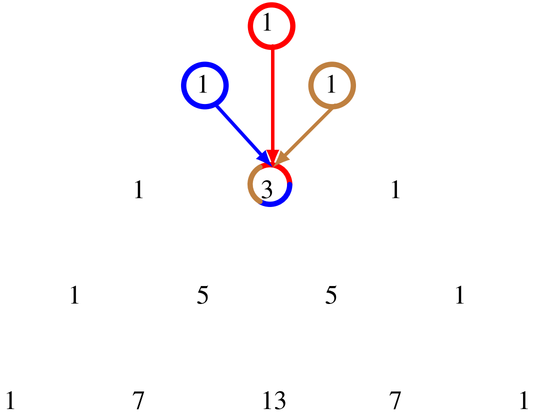

$$D(n,k) = D(n-1,k) + D(n-1,k-1) + D(n-2,k-1).$$

Delannoy numbers

The Delannoy numbers $D(n,k)$ count the number of path from $(0,0)$ to (n,k) using three directions north, north-east or east.

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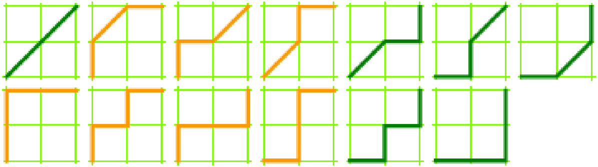
The number of path from $(0,0)$ to $(2,2)$ is given by $D(2,2) = 13$



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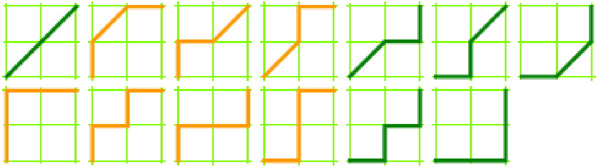

The Delannoy numbers can be expressed in term of binomial coefficients as

$$D(n,k) = \sum_{i=0}^{k} \binom{k}{i} \binom{n+k-i}{k} = \sum_{i=0}^{k} 2^i \binom{k}{i} \binom{n}{i},$$

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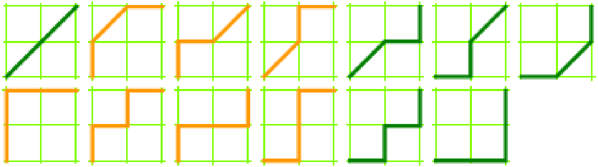

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The generating function $\sum D(n,k) x^n y^k = \dfrac{1}{1-x-y-xy}$.

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where $\binom{n}{k}_s$ known as bi^{s}nomial coefficients also as ordinary multinomial coefficients. The bi^{s}nomial coefficients satisfy the recurrence relation, for $n \geq 1$

$$\binom{n}{k}_s = \binom{n-1}{k}_s + \binom{n-1}{k-1}_s + \cdots + \binom{n-1}{k-s}_s,$$

with $\binom{0}{0} = 1$ and $\binom{n}{k}_s = 0$ for $k < 0$ or $k > sn$.

Bi^{2}nomial coefficients triangle

For $s = 2$

Bi^{2}nomial coefficients triangle

For $s = 2$

1												
1	1	1										
1	2	3	2	1								
1	3	6	7	6	3	1						
1	4	10	16	19	16	10	4	1				
1	5	15	30	45	61	45	30	15	5	1		
1	6	21	50	90	136	151	136	90	50	21	6	1

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Bi^{3}nomial coefficients triangle

For $s = 3$

Bi^{3}nomial coefficients triangle

For $s = 3$

1														
1	1	1	1											
1	2	3	4	3	2	1								
1	3	6	10	12	12	10	6	3	1					
1	4	10	20	31	40	44	40	31	20	10	4	1		
1	5	15	35	65	101	135	155	135	101	65	35	15	5	1

Bi^{3}nomial coefficients triangle

For $s = 3$

1														
1	1	1	1											
1	2	3	4	3	2	1								
1	3	6	10	12	12	10	6	3	1					
1	4	10	20	31	40	44	40	31	20	10	4	1		
1	5	15	35	65	101	135	155	135	101	65	35	15	5	1

Bi^{3}nomial coefficients triangle

For $s = 3$

1														
1	1	1	1											
1	2	3	4	3	2	1								
1	3	6	10	12	12	10	6	3	1					
1	4	10	20	31	40	44	40	31	20	10	4	1		
1	5	15	35	65	101	135	155	135	101	65	35	15	5	1

Bi^{3}nomial coefficients triangle

For $s = 3$

1														
1	1	1	1											
1	2	3	4	3	2	1								
1	3	6	10	12	12	10	6	3	1					
1	4	10	20	31	40	44	40	31	20	10	4	1		
1	5	15	35	65	101	135	155	135	101	65	35	15	5	1

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For $s = 3$

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1	2	3	4	3	2	1								
1	3	6	10	12	12	10	6	3	1					
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1	3	6	10	12	12	10	6	3	1					
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1	2	3	4	3	2	1								
1	3	6	10	12	12	10	6	3	1					
1	4	10	20	31	40	44	40	31	20	10	4	1		
1	5	15	35	65	101	135	155	135	101	65	35	15	5	1

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1	1	1	1											
1	2	3	4	3	2	1								
1	3	6	10	12	12	10	6	3	1					
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1	5	15	35	65	101	135	155	135	101	65	35	15	5	1

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For $s = 3$

1														
1	1	1	1											
1	2	3	4	3	2	1								
1	3	6	10	12	12	10	6	3	1					
1	4	10	20	31	40	44	40	31	20	10	4	1		
1	5	15	35	65	101	135	155	135	101	65	35	15	5	1

Bi^{3}nomial coefficients triangle

For $s = 3$

1														
1	1	1	1											
1	2	3	4	3	2	1								
1	3	6	10	12	12	10	6	3	1					
1	4	10	20	31	40	44	40	31	20	10	4	1		
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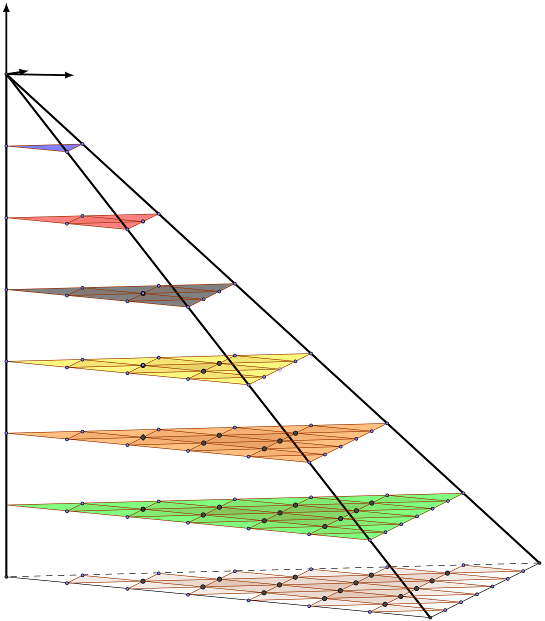
Coefficients trinomiaux



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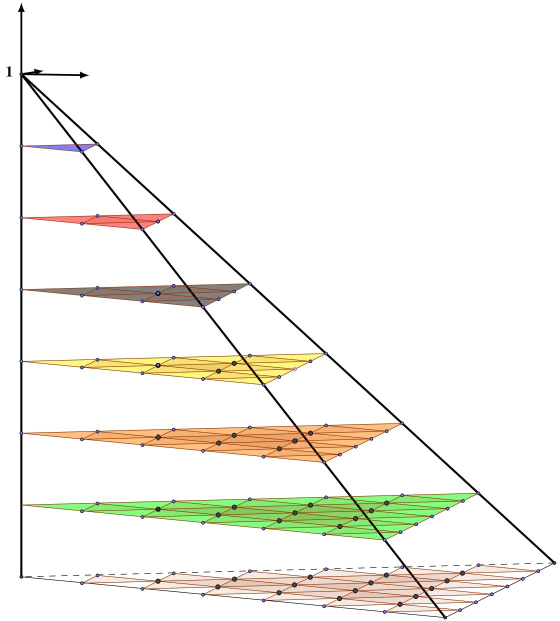

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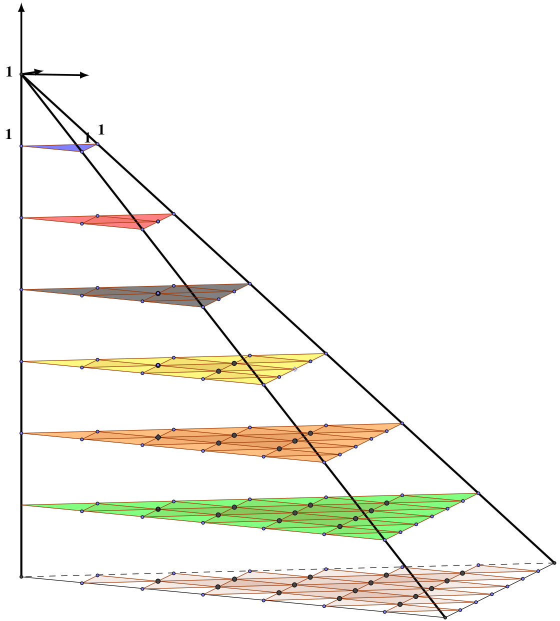

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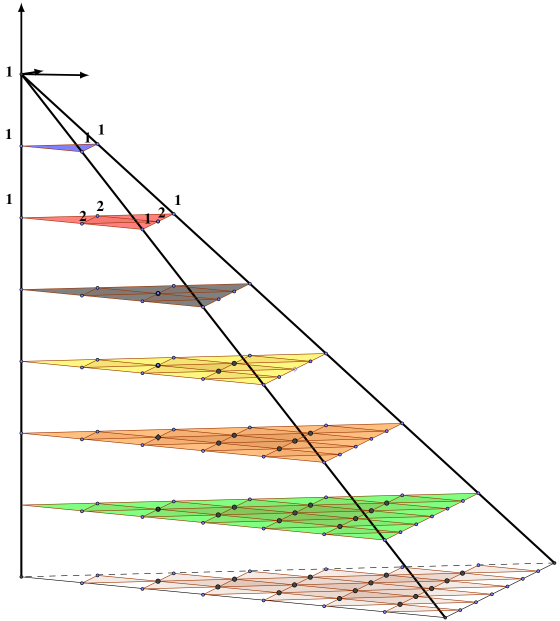

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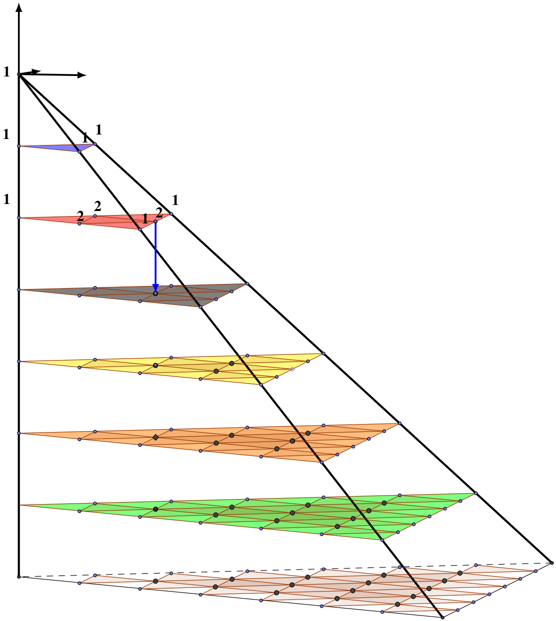

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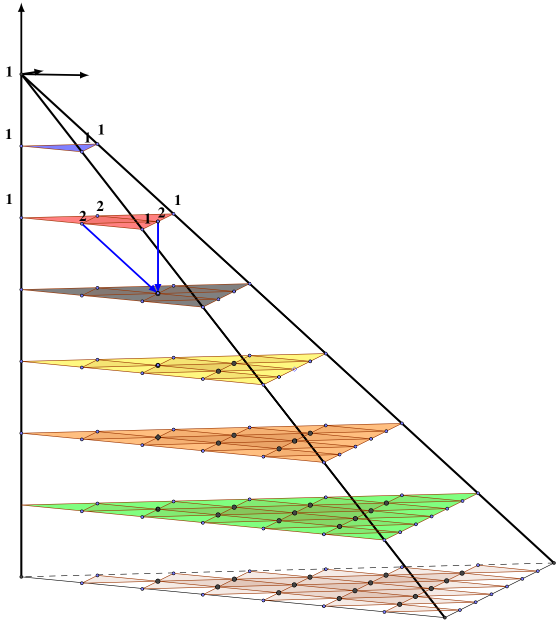

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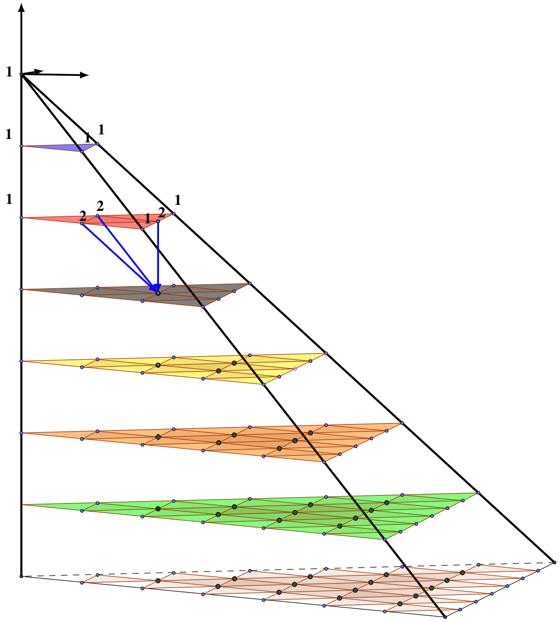

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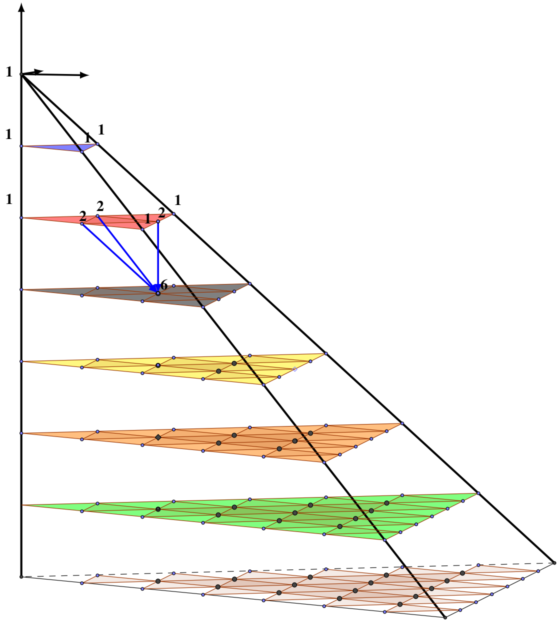

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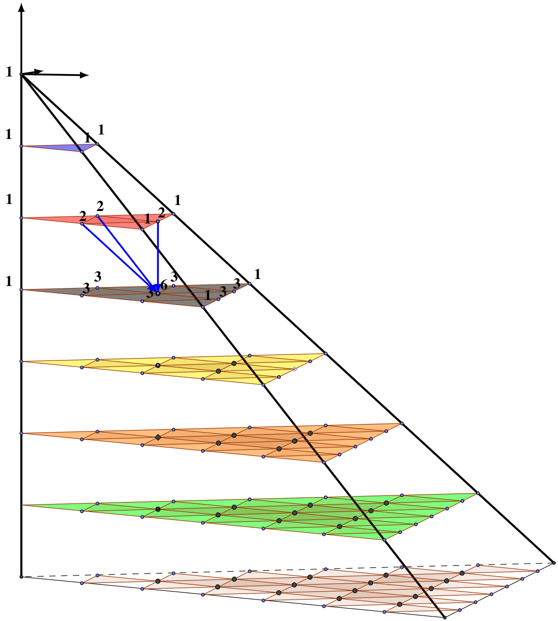

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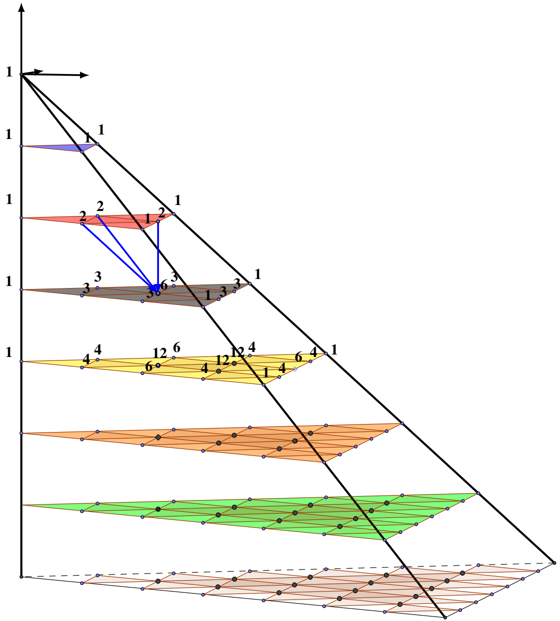

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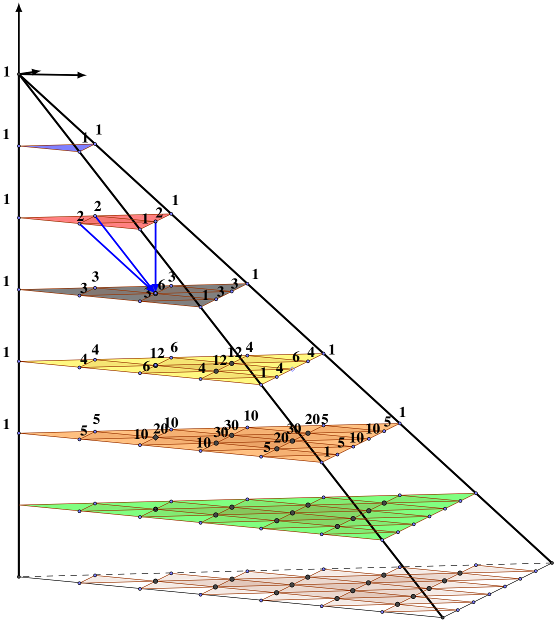

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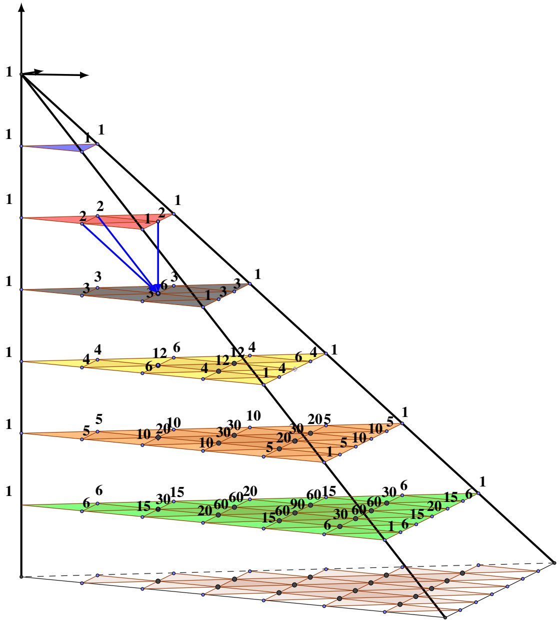

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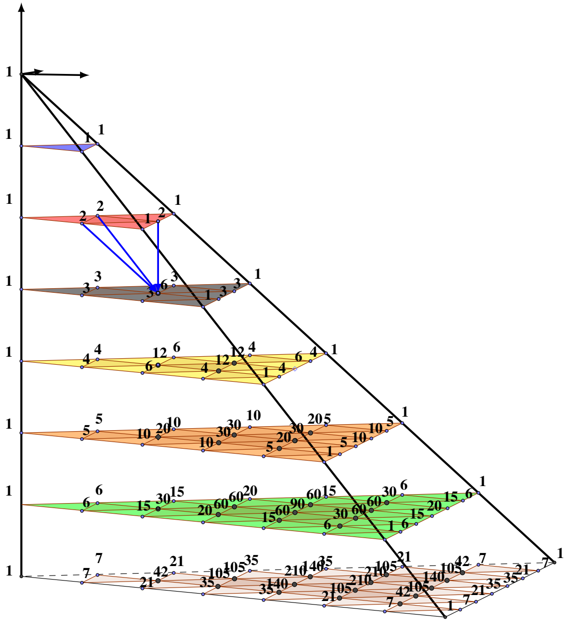
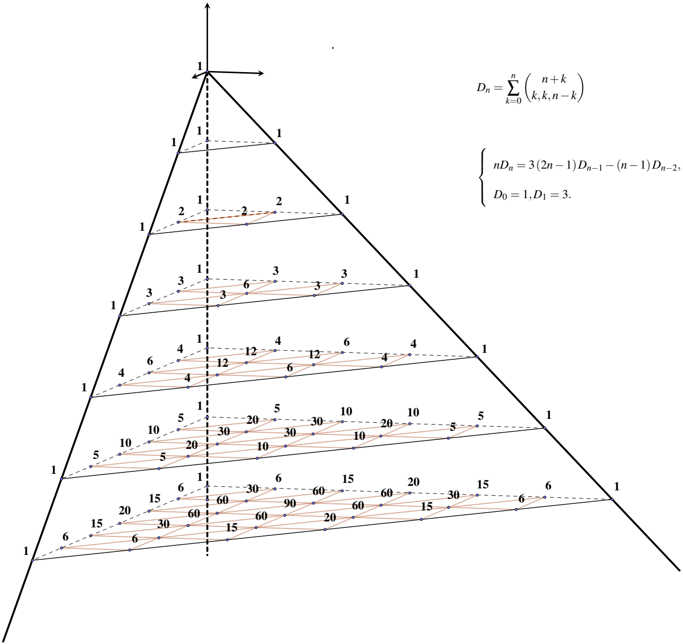
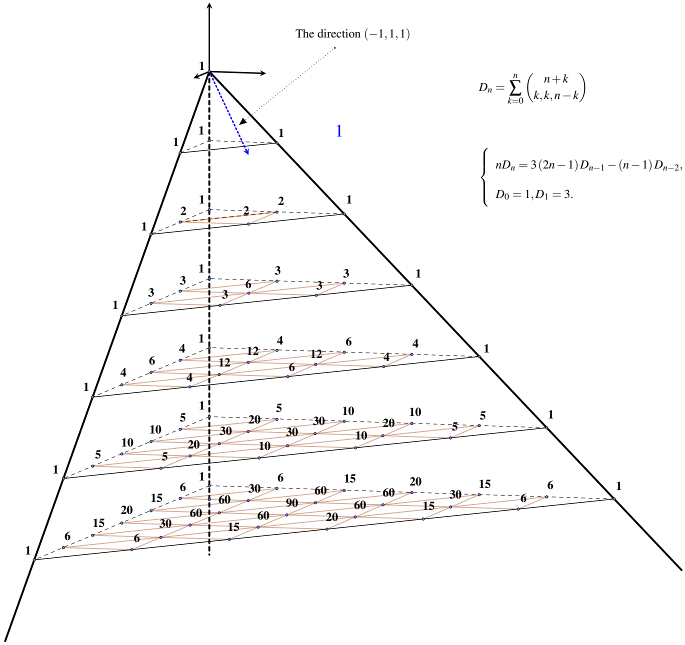




$$D_n = \sum_{k=0}^{n} \binom{n+k}{k,k,n-k}$$

$$\begin{cases} nD_n = 3\,(2n-1)\,D_{n-1} - (n-1)\,D_{n-2}, \\ D_0 = 1, D_1 = 3. \end{cases}$$



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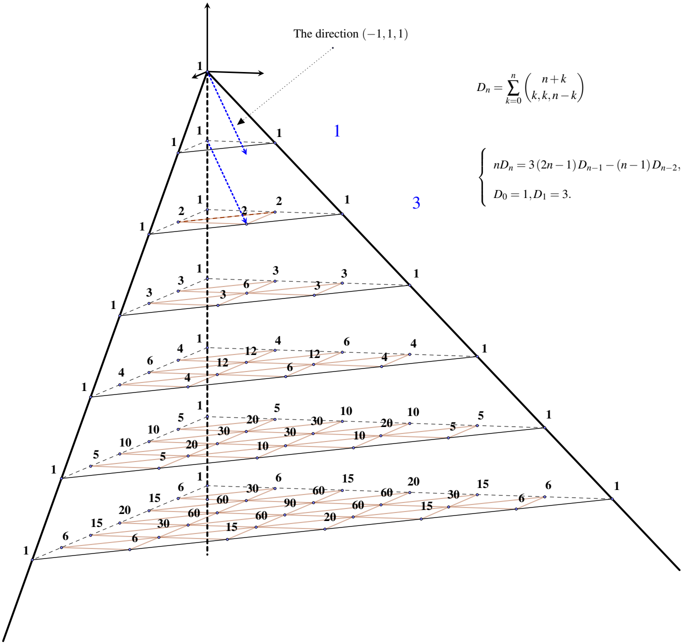

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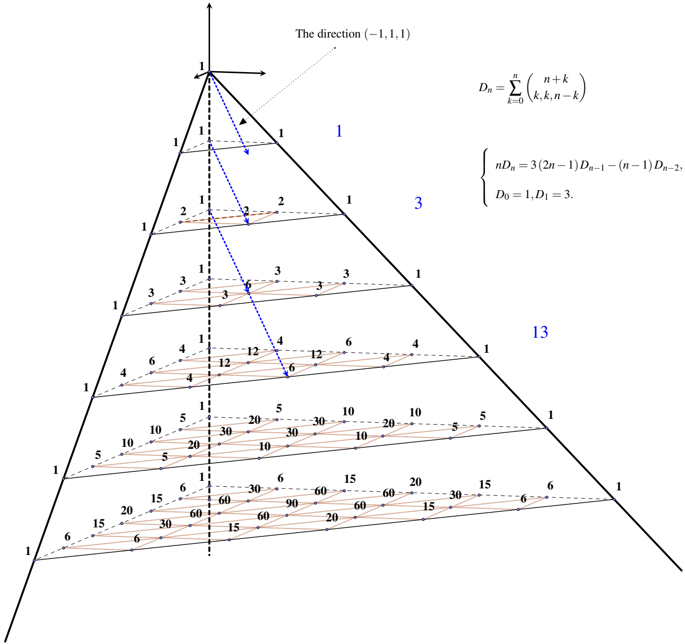

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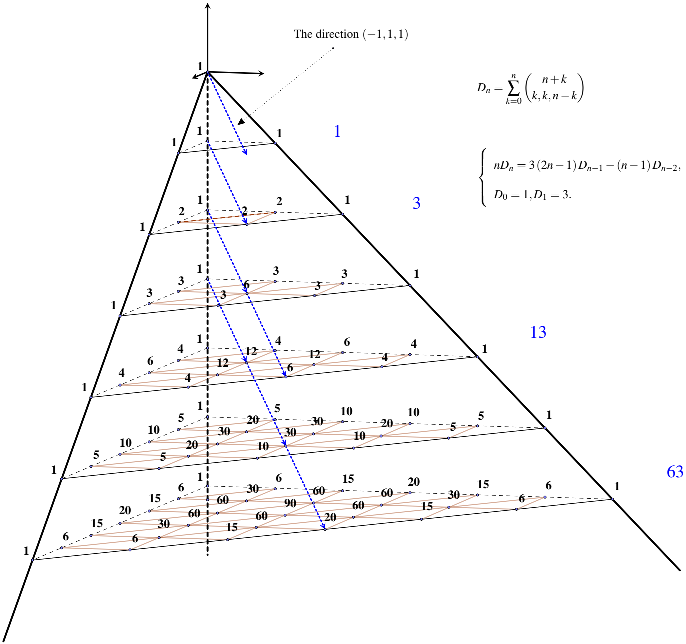

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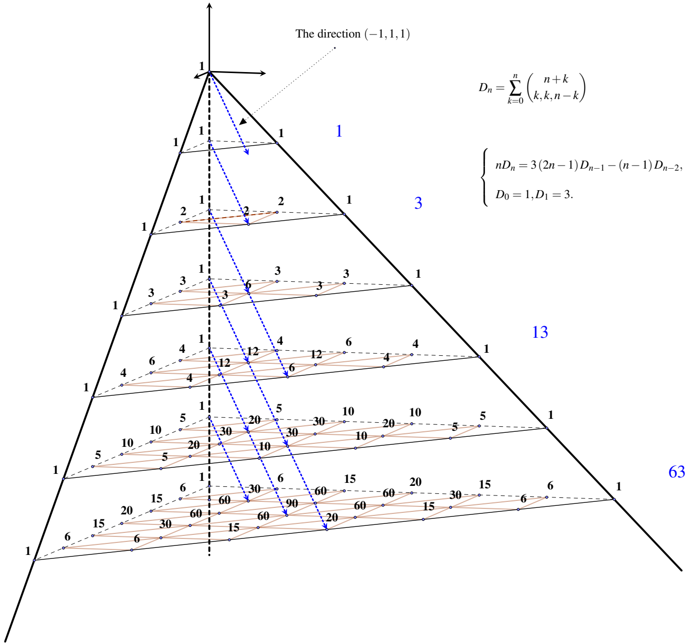

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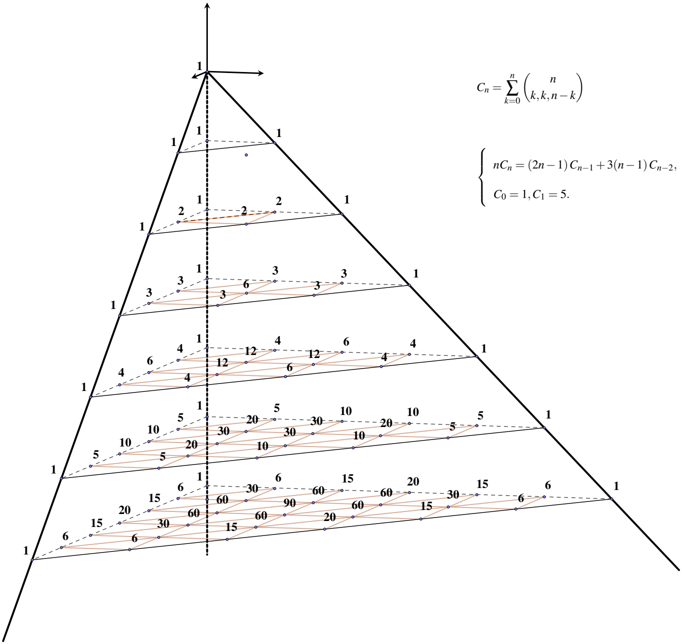

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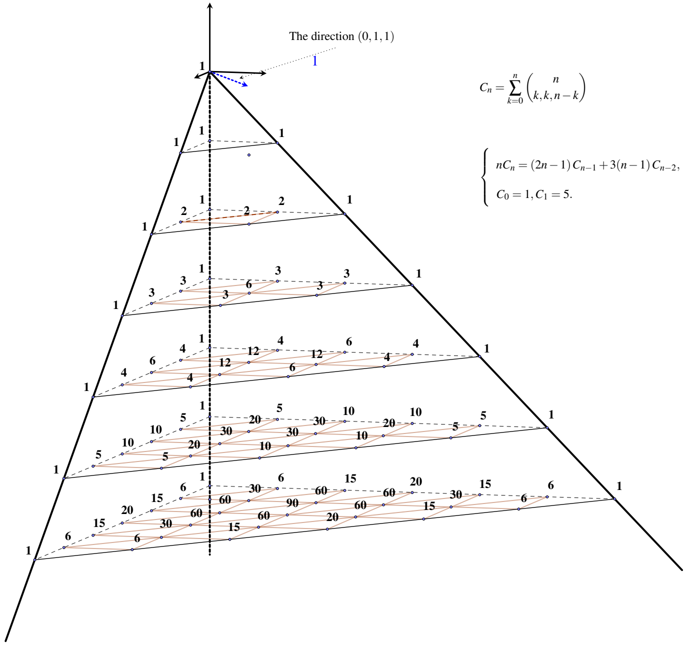
$$\begin{cases} nC_n = (2n-1)\,C_{n-1} + 3(n-1)\,C_{n-2}, \\ C_0 = 1, C_1 = 5. \end{cases}$$



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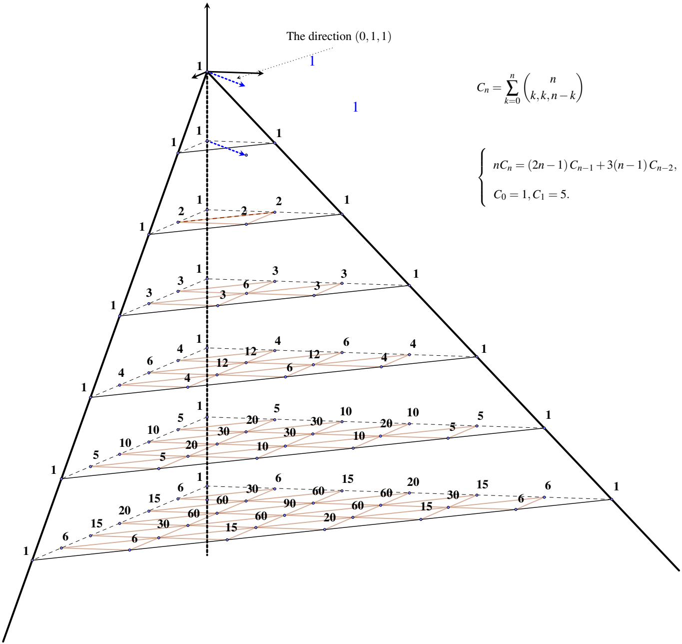

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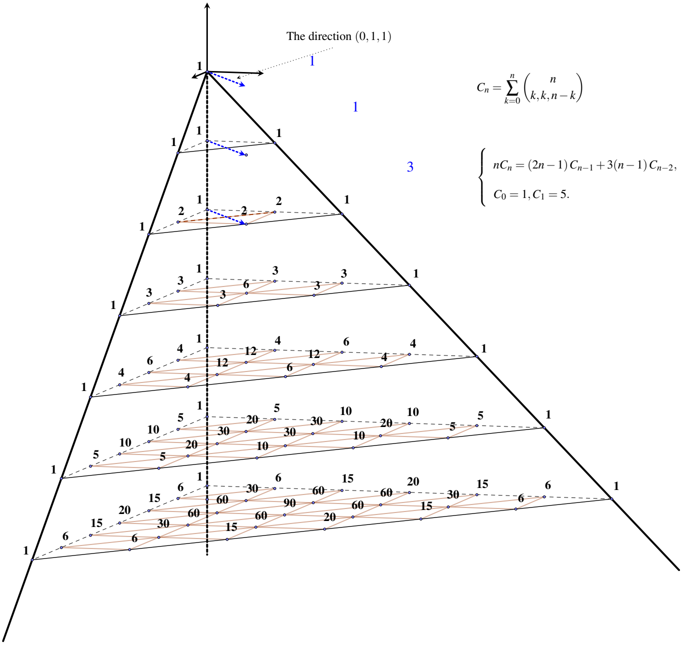

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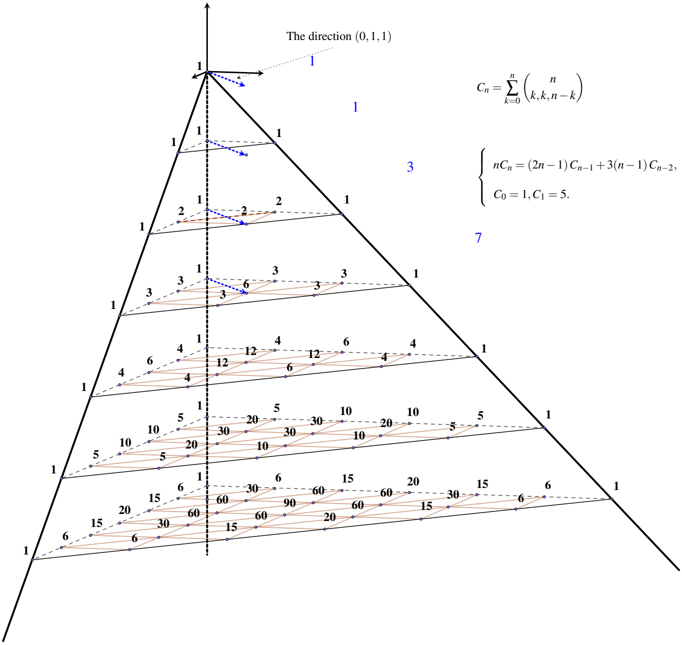

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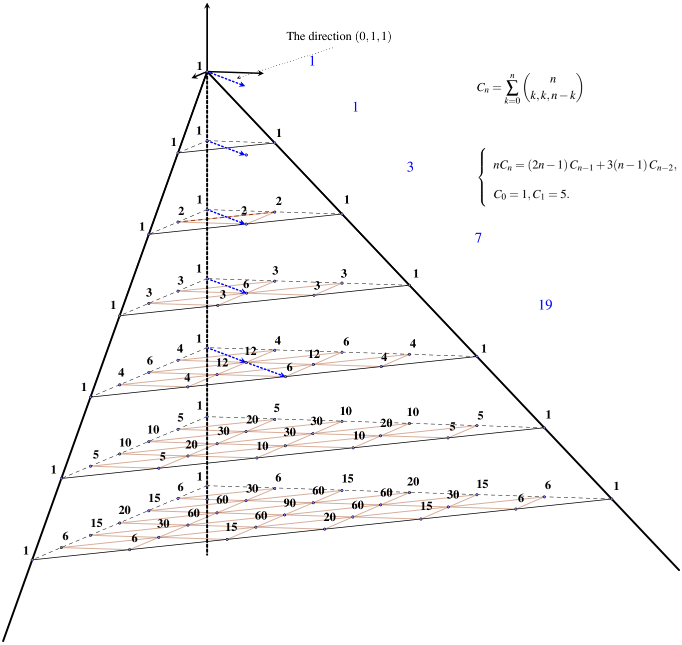

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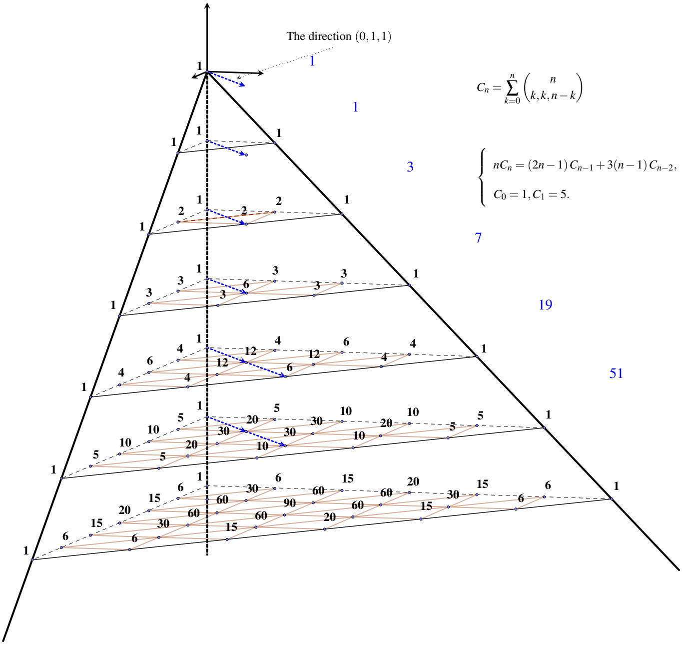

The direction $(0,1,1)$
1
1
3
7
19
51
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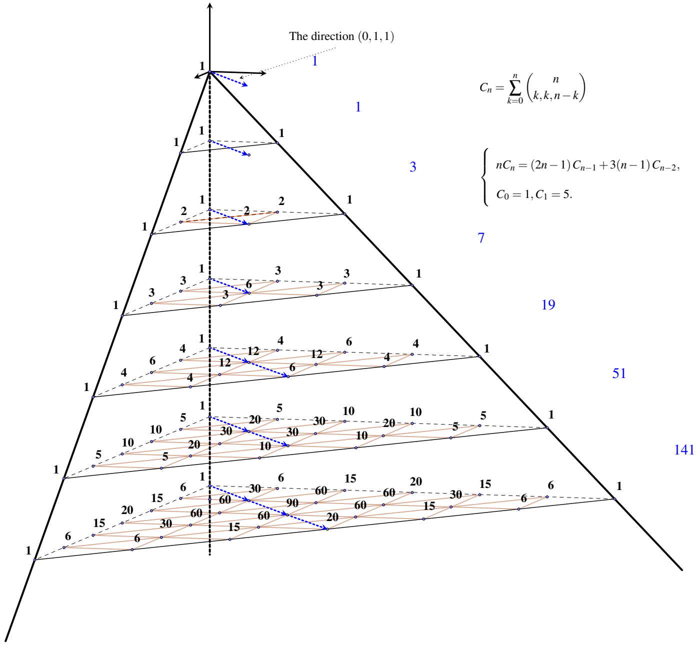

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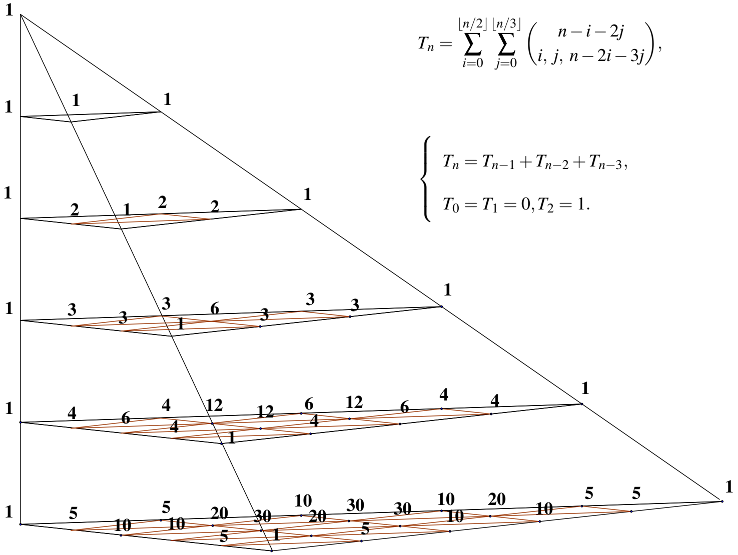

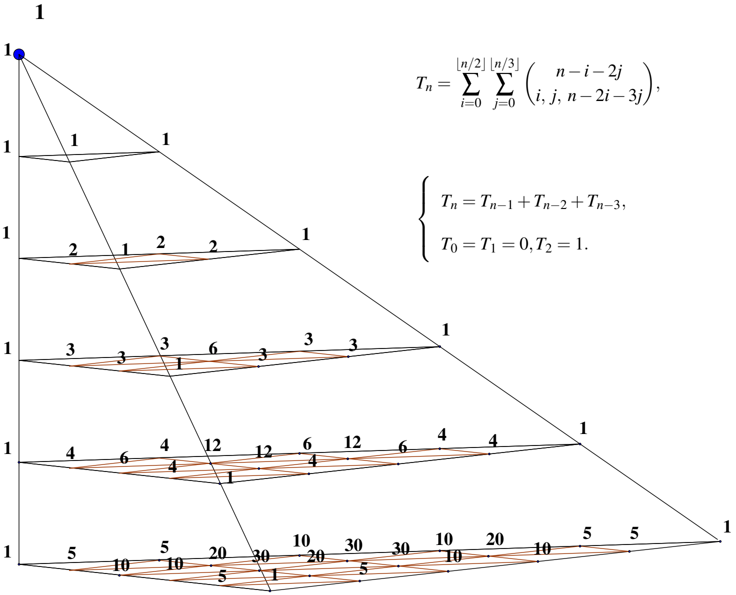

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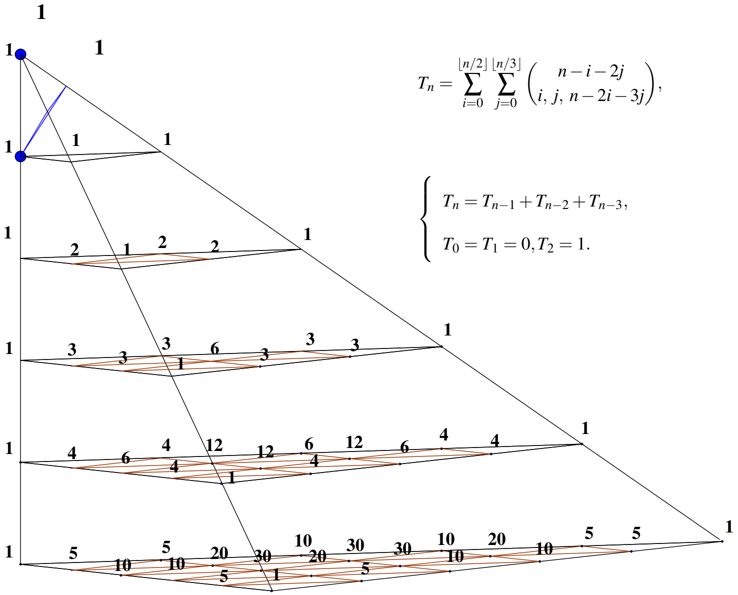

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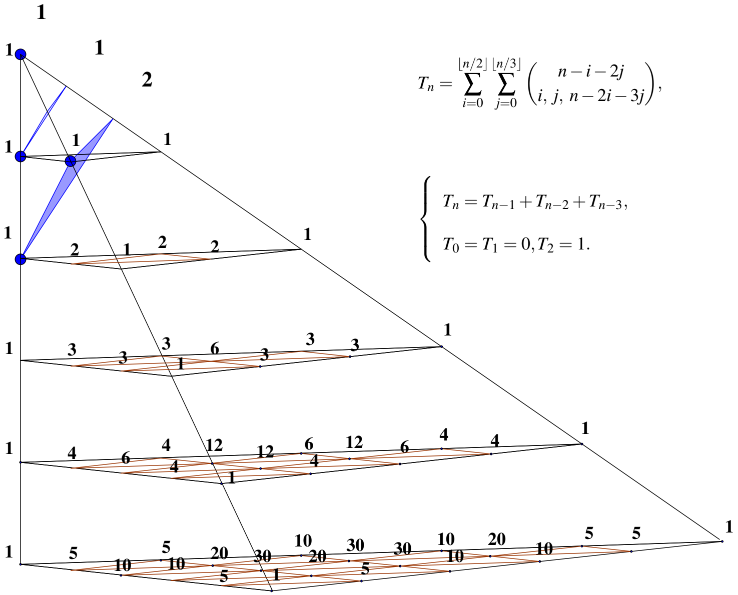

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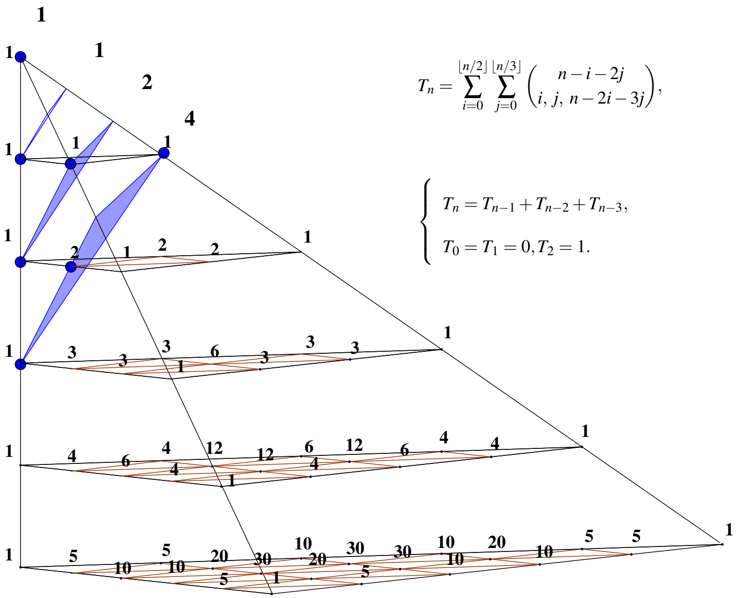

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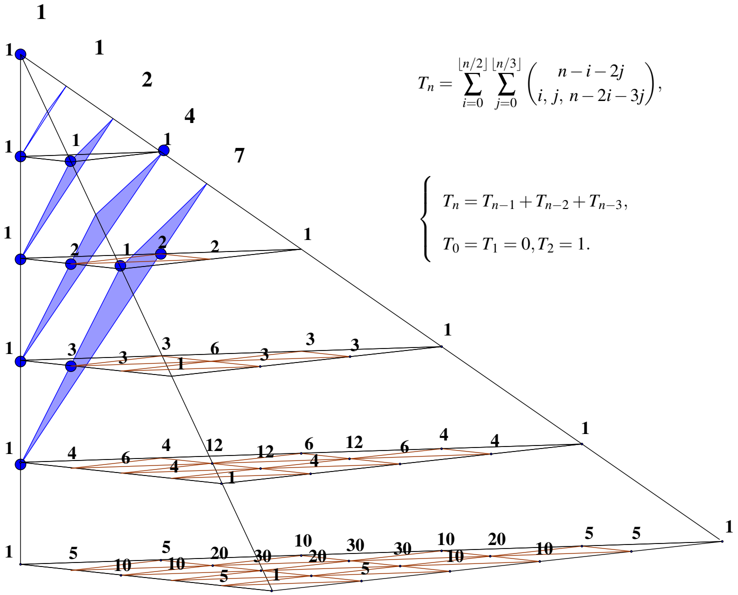

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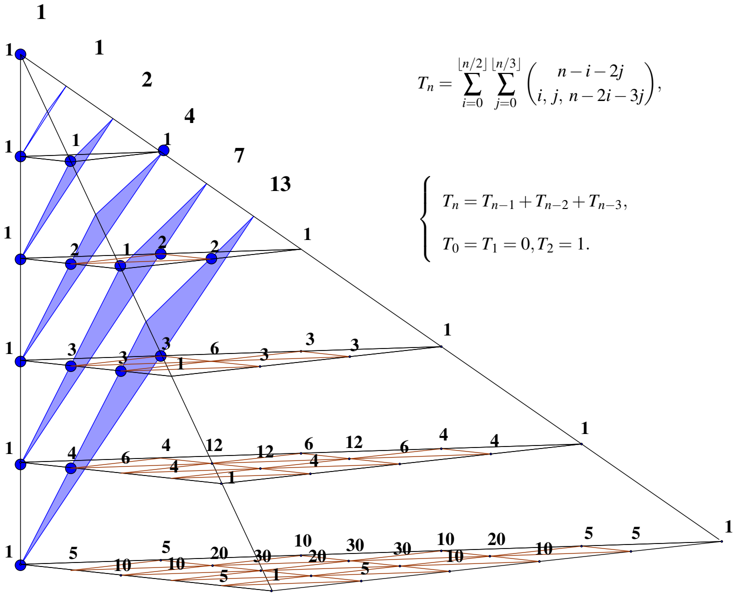

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Hyperbolic Pascal triangle

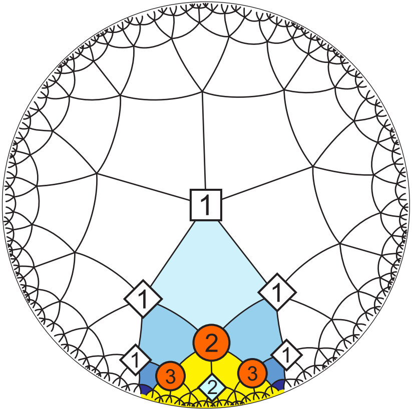

Figure: Hyperbolic Pascal triangle on the mosaïc {4,5}

In the mosaïc of type $\{p,q\}$, p count the number of edges around a cell and q count the number of edges related to a vertex.

Hyperbolic Pascal triangle

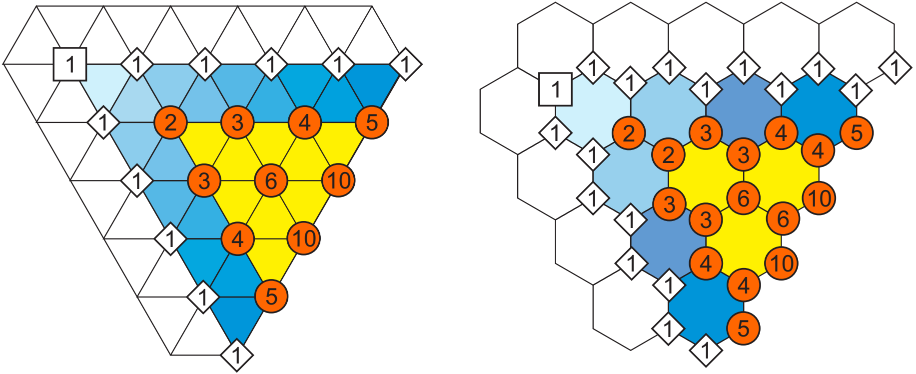

Figure: Pascal triangle on the euclidien mosaïcs $\{3, 6\}$ et $\{6, 3\}$

Hyperbolic Pascal triangle

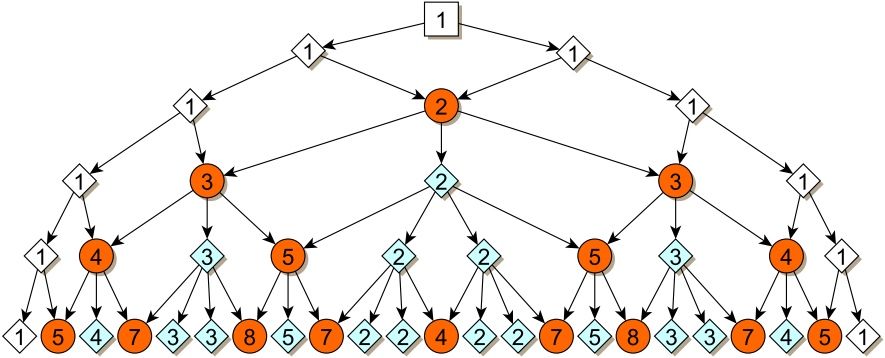

Figure: First layers of hyperbolic Pascal triangle $\{4,5\}$