Elementary differential geometry

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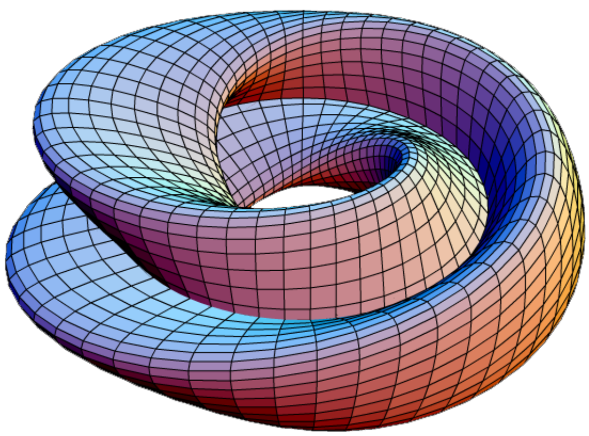

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Abstract

Differential geometry is a very interesting branch of mathematics. Roughly speaking, is the mathematical apparatus that uses generalizations of calculus in order to understand geometric shapes, sizes and curvature of a space. Maybe one of the most interesting things in this theory is the notion of curvature. It turns out that we can talk about the curvature of a space from a purely intrinsic point of view, i.e. without reference to an ambient space. This was one of Gausss many great breakthroughs a few centuries ago. Subsequently, Bernhard Riemann generalized Gauss curvature of a surface to spaces of higher dimensions. So, one can actually talk about the curvature of a higher dimensional space, which is incredibly useful.

As one may suspect, these mathematical things are absolutely crucial from a physical point of view. If we think about the great work that Einstein did, one of the things that he needed was a mathematical framework which described the curvature of four-dimensional spacetime, because gravitation manifests itself as curvature of space and time. This is the language of differential geometry.

Mainly it studies geometric problems using analytic tools such as differential calculus, integrals...

Historically, differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces. Mathematical analysis of curves and surfaces had been developed to answer some of the nagging and unanswered questions that appeared in calculus, like the reasons for relationships between complex shapes and curves, series and analytic functions. These unanswered questions indicated greater, hidden relationships. The general idea of natural equations for obtaining curves from local curvature appears to have been first considered by Leonhard Euler in 1736, and many examples with fairly simple behavior were studied in the 1800s. When curves, surfaces enclosed by curves, and points on curves were found to be quantitatively, and generally, related by mathematical forms, the formal study of the nature of curves and surfaces became a field of study in its own right, with Monge's paper in 1795, and especially, with Gauss's publication of his article, titled "Disquisitiones Generales Circa Superficies Curvas", in Commentationes Societatis Regiae Scientiarum Gottingesis Recentiores in 1827. Initially applied to the Euclidean space, further explorations led to non-Euclidean space, and metric and topological spaces.

This course gives an introduction to this domain, it doesn't requires any background in this mathematical discipline. This course is written for L3 students.

Our main references are [**Spi99**], [**RS18**],[**Oan18**], [**Sch08**], [**Gua18**].

Contents

NOTATIONS: Through out this work, we use the following general notations:

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ for the set of nonnegative integers, integers, the fields of rational numbers, real numbers, complex numbers respectively.
- Maps are generally denoted by $f, g, h....$
- Integers are denoted by $k, n, m....$
- We say that a function f is C^k if it is $k-$times differentiable and its $k^{th}-$differential is continuous.
- The differential of a function f at p is denoted $d_p f$.
- The Jacobian of a function f at p is denoted $J_p(f)$.

CHAPTER 1

Differential calculus

Let $U, V \subset \mathbb{R}^n$ open subsets, $k \in \mathbb{N}$. Recall that a function $f : U \to \mathbb{R}^m$ is said to be of class C^k if f is $k-$times differentiable and its k^{th} differential is continuous. A function $f : U \to V$ is said C^k-diffeomorphism if f is bijective of class C^k and its inverse is again of class C^k. U and V are said to be C^k-diffeomorphic, denoted $U \cong V$, if there is a C^k-diffeomorphism $f : U \to V$. A function of class C^∞ is called smooth.
A function $f : U \to \mathbb{R}^n$ is called a local C^k-diffeomorphism at $x_0 \in U$ if there exist neighborhoods V and W of x_0 and $f(x_0)$ respectively such that $f|_V : V \to W$ is a C^k-diffeomorphism. The function f is said a local C^k-diffeomorphism on U if it is a local C^k-diffeomorphism at x_0 for any $x_0 \in U$.

Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be two open subsets. If $f : U \to V$ is a diffeomorphism, then the Jacobian matrix $d_p f \in \mathbb{R}^{m\times n}$ is invertible for every $p \in U$, in particular $m = n$. This can be seen using the chain rule for differential operator.

Remark 0.1. A differentiable bijection is not necessarily a diffeomorphism. $f(x) = x^3$, for example, is not a diffeomorphism from $\mathbb{R}$ to itself because its derivative vanishes at 0 (and hence its inverse is not differentiable at 0). This is an example of a homeomorphism that is not a diffeomorphism.

Proposition 0.2. *Let $U, V \subset \mathbb{R}^n$ and $f : U \to V$ be a C^1-diffeomorphism, then for any $x_0 \in U$, we have $d_{x_0} f : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism and*

$$(d_{x_0} f)^{-1} = d_{f(x_0)}(f^{-1})$$

Proof. Use $f \circ f^{-1} = \mathrm{id}$. □

Proposition 0.3. *A local C^k-diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ that is bijective is a global C^k-diffeomorphism.*

Proof. We need just to proof that f^{-1} is C^k. Let $y \in \mathbb{R}^n$ such that $y = f(x)$. Since f is local C^k-diffeomorphism then by definition f^{-1} is C^k near y. □

Definition 0.4. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function.

(1) f is called *immersion* if for each $x \in \mathbb{U}$, the function $d_x f$ is injective.
(2) f is called *submersion* if for each $x \in \mathbb{U}$, the function $d_x f$ is surjective.
(3) f is called *embedding* if f is *immersion* and if f is homeomorphism from U to $f(U)$.

Recall now some general facts.

Theorem 0.5 (Young's Theorem). *If $f : U \to \mathbb{R}^m$ is defined on the open set $U \subset \mathbb{R}^n$ and is of class C^2, then $\partial_i \partial_j f = \partial_j \partial_i f$ for any $1 \leqslant i, j \leqslant n$.*

We recall the following important result from topology.

THEOREM 0.6 (Invariance of domain). *Let U be an open subset of $\mathbb{R}^n$, and $f : U \to \mathbb{R}^n$ a continuous and injective map. Then $f(U)$ is open, and $f : U \to f(U)$ is a homeomorphism.*

1. Inverse function theorem

Now, we come to the inverse function theorem.

THEOREM 1.1. *Let $U \subset \mathbb{R}^n$ open and $p \in U$. Let $f : U \to \mathbb{R}^n$ a function of class C^k and assume that $d_p f$ is invertible. Then there exist open sets V and W containing p and $f(p)$ respectively such that the restriction of f on V is a bijection onto W with a C^k-inverse. Moreover, we have*

$$d_y(f^{-1}) = [d_{f^{-1}(y))} f]^{-1}.$$

PROOF. See [**Ma09**, Théorème 33.9] □

REMARK 1.2. The inverse function theorem states that a function f is a diffeomorphism if and only if it is invertible, and for all $p \in U$, the Jacobian matrix $J_p f$ is invertible. (That is, one does not actually have to check smoothness of the inverse map).

EXAMPLE 1.3. The easiest case is in dimension one, we learned that if a function f is continuously differentiable on $]a, b[$ with non-vanishing f', it is either strictly increasing or decreasing so that its global inverse exists and is again continuously differentiable.

2. Implicit function theorem

We consider now a relation between two variables in the form $f(x, y) = 0$. In the best cases we can express explicitly one variable in a function of the other. This is obviously not always possible. The implicit function theorem tells us when this formula does exist without giving it explicitly.
So let $f : U \to \mathbb{R}^m$ be a C^k function, where $U \subset \mathbb{R}^n \times \mathbb{R}^m$.

THEOREM 2.1. *Let $(p, q) \in U$ such that $f(p, q) = 0$ and $d^2_{(p,q)} f$ is invertible, where $d^2_{(p,q)} f$ is the differential of f with respect to the second term $\mathbb{R}^m$. Then there exist an open set $V_1 \times V_2$ in U containing (p, q) and a C^k-map $\phi : V_1 \to V_2$ with $\phi(p) = q$, such that $f(x, \phi(x)) = 0\ \forall x \in V_1$. Moreover, if ψ is another C^k-map in some open set containing p to V_2 satisfying $f(x, \psi(x)) = 0$ and $\psi(p) = q$, then ψ coincides with ϕ in their common set of definition.*

PROOF. We will use the inverse function theorem for the function

$$g : U \to \mathbb{R}^n \times \mathbb{R}^m$$

$$(x, y) \mapsto (x, f(x, y)).$$

We have $g = (p|_U, f)$, where $p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the first projection. So g is C^k-function.
Step 1: We show that $d_{(p,q)} g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ is an isomorphism, i.e. injective. So let (x, y) be in the kernel, then we have

$$d_{(p,q)} g(x, y) = (x, d_{(p,q)} f(x, y)) = (0, 0),$$

so $x = 0$, and $d_{(p,q)}f(0, y) = d^1_{(p,q)}f(0) + d^2_{(p,q)}f(0, y) = d^2_{(p,q)}f(0, y) = 0$, since by hypothesis $d^2_{(p,q)}f$ is invertible we deduce that $y = 0$.

Step 2: Since $d_{(p,q)}g$ is invertible, we can use the inverse function theorem for g, there exists an open neighborhood U of (p, q) and a neighborhood V' of $g(p, q)$ such that $g : U \to V'$ is a C^k-diffeomorphism. There exist neighborhoods $U_1 \subset \mathbb{R}^n$ of x_0 and $U_2 \subset \mathbb{R}^m$ of q such that $U_1 \times U_2 \subset U$. So g is a C^k-diffeomorphism $U_1 \times U_2 \to V := g(U_1 \times U_2) \subset V'$. Note that V is open in $\mathbb{R}^n \times \mathbb{R}^m$. And we have $g^{-1} = (h_1, h_2) : V \to U_1 \times U_2$ such that $h_i : V \to U_i$ are C^k and $h_1(\alpha, \beta) = \alpha$. Indeed

$$(\alpha, \beta) = g(g^{-1}(\alpha, \beta)) = g(h_1(\alpha, \beta), h_2(\alpha, \beta)) = (h_1(\alpha, \beta), f(h_1(\alpha, \beta), h_2(\alpha, \beta))).$$

Step 3: Let φ given by

$$\varphi = h_2 \circ \iota,$$

where $\iota : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m; x \mapsto (x, 0)$..

Now let $V_p = \iota^{-1}(V)$ and $V_q = U_2$. So V_p and V_q are neighborhood of p and q respectively and $\varphi : V_p \to V_q$ is a C^k-function. Moreover, for any $(x, y) \in V_p \times V_q$

$$\begin{aligned} f(x, y) = 0 &\Leftrightarrow g(x, y) = (x, 0) \\ &\Leftrightarrow (x, y) = g^{-1}(x, 0) = (h_1(x, 0), h_2(x, 0)) = (x, h_2(x, 0)) \\ &\Leftrightarrow y = h_2(x, 0) \\ &\Leftrightarrow y = \varphi(x) \end{aligned}$$

which ends the proof. □

The following proposition gives explicitly the differential of the implicit function.

Proposition 2.2. *Under the same hypothesis of the implicit function theorem we have*

$$d_p\phi = -(d^2_{(p,q)}f)^{-1} \circ d^1_{(p,q)}f,$$

where d^1 (resp. d^2) is the differential with respect to the first variable p (resp. second variable q).

Proof. Just differentiate the formula $f(x, \phi(x)) = 0$. □

3. Constant rank theorem

Let $x_0 \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function at x_0. The rank of f at x_0 is by definition the rank of the linear morphism $d_{x_0}f : \mathbb{R}^n \to \mathbb{R}^m$, which is by definition the dimension of $d_{x_0}f(\mathbb{R}^n)$, we denote it $\mathrm{rank}_{x_0}f$. We give now the notion of conjugation.

Definition 3.1.

(1) We say that two functions $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are C^k-conjugated if there exist two C^k-diffeomorphisms $\phi, \psi : \mathbb{R}^n \to \mathbb{R}^m$ such that $\psi \circ f = g \circ \phi$. That's the following diagram commutes

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[d, "\phi"'] & \mathbb{R}^m \arrow[d, "\psi"] \\ \mathbb{R}^n \arrow[r, "g"] & \mathbb{R}^m. \end{tikzcd}$$

(2) We say that two functions $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are locally C^k-conjugated near $x_0 \in \mathbb{R}^n$ if there exist two neighborhoods $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ of x_0 and $f(x_0)$ respectively and two open sets $\tilde{U} \subset \mathbb{R}^n$ and $\tilde{V} \subset \mathbb{R}^m$ such that $f : U \to V$ and $g : \tilde{U} \to \tilde{V}$ are C^k-conjugated.

THEOREM 3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a C^k-function, and $x_0 \in \mathbb{R}^n$. Then the following are equivalent*

- *There exists a neighborhood U of x_0 on which f has a constant rank.*
- *f is C^k-conjugated to a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.*

For the proof we refer to [**Ma09**, Théorème 33.34].

CHAPTER 2

Submanifolds of $\mathbb{R}^n$

The first example, and the one to keep in mind, of a submanifold of an Euclidean space is a vector subspace, for example $\mathbb{R}^p \times \{0\}$ contained in $\mathbb{R}^n$. We will define a general submanifold as obtained by local diffeomorphisms from such example, and give equivalent characterizations. recall that the graph of an application $f : U \to V$ is the subset of $U \times V$ formed of pairs $(x, f(x))$ for $x \in U$.

Definition 0.1. Let $M \subset \mathbb{R}^n$ be a subset. We say that M is a submanifold of dimension d ($\leqslant n$) and of class C^k if for all x in M, there is a neighborhood U of x in $\mathbb{R}^n$, a neighborhood V of 0 in $\mathbb{R}^n$ and a C^k-diffeomorphism $\phi : U \to V$ such that $\phi(U \cap M) = V \cap (\mathbb{R}^d \times \{0\})$.

There are several equivalent definitions, the above one called definition by local covering. The diffeomorphism ϕ is called a coordinate chart. It is inverse ϕ^{-1} is called a parametrization.
If the dimension $d = 1, 2$, the submanifold M is called curve, surface respectively. If $d = n - 1$, M is called *hypersurface* in $\mathbb{R}^n$. Note also that M inherits a topology from the ambient space $\mathbb{R}^n$.

Example 0.2. Any vector subspace of $\mathbb{R}^n$ is a submanifold. Any open subset is a submanifold.

Example 0.3. Consider the 2-sphere

$$M := S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}.$$

Let $U \subset \mathbb{R}^3$ and $V \subset \mathbb{R}^2$ be the open sets

$$U := \{(x, y, z) \in \mathbb{R}^3 | z > 0\},\ V := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}.$$

The map $\phi : U \cap M \to V$ given by $(x, y, z) \mapsto (x, y)$ is bijective and its inverse is given by $\phi^{-1} : (x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$. Since both ϕ and ϕ^{-1} are smooth, the map is a coordinate chart on S^2. Similarly, we can use the open sets $z < 0$, $y > 0$, $y < 0$, $x > 0$, $x < 0$ to cover S^2 by six coordinate charts. Hence S^2 is a manifold. A similar argument shows that the unit sphere $S^m \subset \mathbb{R}^{m+1}$ is a manifold for every integer $m > 0$.

If M_1 and M_2 are two submanifold of $\mathbb{R}^n$ and $\mathbb{R}^m$ of dimension d_1 and d_2 respectively. Then one can see that $M_1 \times M_2$ is a submanifold of $\mathbb{R}^{n+m}$ of dimension $d_1 + d_2$.

Example 0.4. A particular case is given by

$$\underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$$

called the $n-$torus. It is a manifold of dimension n inside $\mathbb{R}^{n+1}$.

Recall the following definition

DEFINITION 0.5. Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^m$ be a smooth function. An element $c \in \mathbb{R}^m$ is called a *regular value* of f if, for all $p \in U$, we have

$$f(p) = c \Longrightarrow d_p f : \mathbb{R}^n \to \mathbb{R}^m \text{ is surjective.}$$

We have

PROPOSITION 0.6. *If c is a regular value of $f : U \to \mathbb{R}^m$, then $f^{-1}(c)$ is a $n - m$ dimensional smooth submanifold of $\mathbb{R}^n$.*

PROOF. See Exercise 0.7. □

1. Tangent bundle

Let $M \subset \mathbb{R}^n$ be a smooth $m-$dimensional manifold and fix a point $p \in M$.

DEFINITION 1.1. A vector $v \in \mathbb{R}^n$ is called a *tangent vector* of M at p if there exists a smooth curve $\gamma :]-1, 1[\to M$ such that

$$\gamma(0) = p,\ \gamma'(0) = v.$$

The set of tangent vectors to M at p is called the linear tangent space of M at p and it is denoted T_pM. The affine tangent space of M at p is the translate of T_pM with the vector p.

PROPOSITION 1.2. *T_pM is actually a linear subspace of $\mathbb{R}^n$. The dimension of T_pM equals the dimension of M.*

PROOF. Let $U \subset M$ be an open set containing p and let $\phi : U \to V \subset \mathbb{R}^d$ be a diffeomorphism onto an open subset. Let $q := \phi(p)$ and let $\psi := \phi^{-1}$ be the inverse map. We claim the following which induces the proposition

$$T_pM = d_q\psi(\mathbb{R}^d).$$

The first inclusion $d_q\psi(\mathbb{R}^d) \subset T_pM$. Let $r > 0$, such that the ball $B_r(q) \subset V$. Let $v \in \mathbb{R}^m$ and let $\varepsilon > 0$ such that $\varepsilon|v| < r$. Hence for all $t \leqslant \varepsilon$ we have

$$q + tv \in V.$$

Consider the curve $\gamma : (-\varepsilon, \varepsilon) \to M$ defined by

$$\gamma(t) = \psi(q + tv).$$

Since ψ is a diffeomorphism, γ is a smooth curve and we have

$$\gamma(0) = p,\ \gamma'(0) = d_q\psi(v).$$

This gives the first inclusion.
For the inverse inclusion. Let $v \in T_pM$, by definition there exists a smooth curve γ as above such that $\gamma(0) = p$ and $\gamma'(0) = v$. By shrinking the domain of γ, we may assume that the image of γ is contained in U. Let $\tilde{\gamma} = \phi \circ \gamma$. Then $\tilde{\gamma}$ is a smooth curve in V and by using the chain rule, one can get that $d_q\psi(\tilde{\gamma}'(0)) = v$. This proves the claim. □

The collection $TM := \bigcup_{p\in M} T_pM$ is more than a set, it is actually a submanifold of $M \times \mathbb{R}^n$, and that is what we call a *vector bundle*. We call TM the *tangent bundle* of M, this what allows us to transport the theory of differential equations to the level of manifolds (see Section 2)

EXAMPLE 1.3.
- The tangent bundle to $M = \mathbb{R}^n$ is the trivial one, namely $M \times \mathbb{R}^n$.
- The tangent bundle to $2-$sphere $S^2 \in \mathbb{R}^3$ is given at the point $p = (a, b, c) \in S^2$ by
$$T_pS^2 = \{(x, y, z) \in \mathbb{R}^3 | ax + by + cz = 0\}.$$
Namely it is the plane in $\mathbb{R}^3$ whose normal vector is $\overrightarrow{Op}$. The bundle TS^2 is not trivial (see eg. hairy ball theorem).
- If $M = U \subset \mathbb{R}^n$ is an open set. Then $TM = M \times \mathbb{R}^n$.

PROPOSITION 1.4. *Let $p \in M$, $U \subset \mathbb{R}^n$ an open neighborhood of p and let $f : U \to \mathbb{R}^m$ a smooth map such that $0 \in \mathbb{R}^m$ is regular value for f and $U \cap M = f^{-1}(0)$. Then*
$$T_pM = Ker(d_pf) = \{v \in \mathbb{R}^n | d_pf(v) = 0\}.$$

PROOF. If $v \in T_pM$, then there is a smooth curve $\gamma :] - \varepsilon, \varepsilon[\to M$ such that $\gamma(0) = p$ and $\gamma(0) = v$. For t sufficiently small we have $\gamma(t) \in U$, where $f(\gamma(t)) = 0$. Hence
$$d_pf(v) = d_{\gamma(0)}(f(\gamma'(0))) = (f \circ \gamma)'(0) = 0.$$
and this implies $T_pM \subset Ker(d_pf)$. Since T_pM and the kernel of d_pf are both $m-$dimensional linear subspaces of $\mathbb{R}^n$ we deduce that $T_pM = Ker(d_pf)$. $\square$

REMARK 1.5. The definition we have given of T_pM has some disadvantages. In particular, it is not clear at all how to organize T_pM as a linear space. In some literature, the tangent space is defined using derivative, i.e. a tangent element at p is considered as a class of equivalence (equality on neighborhood of p) of derivation $D : C^\infty_p(M) \to \mathbb{R}$, where $C^\infty_p(M)$ is the space of germs of smooth functions at p. Recall that a derivation on $C^\infty_p(M)$ is a *linear* map $D : C^\infty_p(M) \to \mathbb{R}$, such that for any $f, g \in C^\infty_p(M)$:
$$D(fg) = D(f)g(p) + f(p)D(g).$$
The correspondence between the two is given by associating to a tangent vector v the directional derivative operators D_v (see below) with respect to the vector v.

Derivative. Let $M \subset \mathbb{R}^k$, $N \subset \mathbb{R}^l$ be two submanifolds and let $f : M \to N$ be a map. Then we can define the differential of f (if it exists) generalizing the classical case. The differential of f at $p \in M$ is then a linear map $d_pf : T_pM \to T_{f(p)}N$, defined by associating to a vector $v = \gamma'(0)$ (for some γ) the vector $w = (f \circ \gamma)'(0)$. One can show that this is well defined (i.e. independent from γ) and it gives a linear map. The expression $(f \circ \gamma)'(0)$ is called the directional derivative of f in the direction of v, and it is denoted D_vf.

The differential operator d_p satisfies the chain rule, namely
$$d_p(g \circ f) = d_{f(p)}g \circ d_pf.$$
In particular, if f is a diffeomorphism, then d_pf is an isomorphism.

PROPOSITION 1.6. *Let $f : M \to \mathbb{R}^l$ be a smooth map.*

(i) *The differential d_pf is well defined linear function.*
(ii) *The differential satisfies the chain rule.*

PROOF. (*i*) By taking a neighborhood $U \subset \mathbb{R}^k$, we can find a smooth function $g : U \to \mathbb{R}^l$ such that $g|_{U\cap M} = f_{U\cap M}$. Then , for $v \in T_pM$, we have

$$\begin{aligned} d_p g(v) &= d_{\gamma(0)} g(\gamma'(0)) \\ &= (g\circ\gamma)'(0) \\ &= (f\circ\gamma)'(0). \end{aligned}$$

Since the LHS is independent of γ, we get the result. The linearity also follows.

(*ii*) If $g : N \to L$ for some submanifold L, then we have

$$\begin{aligned} d_p(g\circ f)(v) &= (g\circ f\circ\gamma)'(0) \\ &= d_{f(p)} g(d_p f(v)). \end{aligned}$$

□

Assume that $N = \mathbb{R}$. The directional derivative operator $D_v : C_p^\infty(M) \to \mathbb{R}$ in direction of v determine uniquely the vector v. Indeed, by taking the coordinate functions $e_i(x) = x_i$, we find $D_v e_i = v_i$. Hence, we can define the tangent space as the space of derivations $D : C_p^\infty(M) \to \mathbb{R}$.

Note that for $M = \mathbb{R}^k$, we have $D_v f = d_p f(v)$. In particular, we have $D_{x_i} = \dfrac{\partial}{\partial x_i}$.

DEFINITION 1.7. A smooth map $f : M \to N$ is called an *immersion* (resp. *submersion*) if its differential $d_p f : T_pM \to T_{f(p)}N$ is injective (resp. surjective) for every $p \in M$. It is called *proper* if, for every compact subset $K \subset f(M)$, the preimage $f^{-1}(K)$ is compact. The map f is called an *embedding* if it is a proper injective immersion.

A subset $S \subset M$ is called a submanifold of M if S itself is a submanifold of $\mathbb{R}^k$.

PROPOSITION 1.8. (*i*) *If $f : S \to M$ is an embedding then $f(S)$ is a submanifold of M.*

(*ii*) *If $f : M \to N$ is a submersion then $f^{-1}(c)$ is a submanifold of M, for every $c \in f(M)$.*

PROOF. (*i*) We use the Lemma 1.9. Let $q \in S$, denote $p := f(q) \in P$, where $P := f(S)$, and choose a diffeomorphism $F : V \times W \to U$ as in Lemma 1.9. Then set $V \subset S$ is diffeomorphic to an open subset of $\mathbb{R}^n$ (after schrinking V if necessry), the set $U \cap P$ is $P-$open because $U \subset M$ is $M-$open, and we have $U \cap P = \{F(q, 0)|q \in V\} = f(V)$ by the formulas in Lemma 1.9. Hence the map $f : V \to U \cap P$ is a diffeomorphism whose inverse is the composition of the smooth maps $F^{-1} : U \cap P \to V \times W$ and $V \times W \to V : (q, z) \mapsto q$. Hence a $P-$open neighborhood of p is diffeomorphic to an open subset of $\mathbb{R}^n$. Since $p \in P$ was chosen arbitrary, this shows that P is an $n-$dimensional submanifold of M.

(*ii*) It is a consequence of Proposition 0.6.

□

LEMMA 1.9. *Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^l$ be two submanifolds of dimensions m and n respectively and let $f : M \to N$ be an embedding. Let $q_0 \in M$, and define*

$$P := f(M),\ p_0 := f(q_0) \in P.$$

Then there exists an $N-$open neighborhood $U \subset N$ of p_0, an $M-$open neighborhood $V \subset M$ of q_0, an open neighborhood $W \subset R^{n-m}$ of the origin, and a diffeomorphism $F : V \times W \to U$ such that, for all $q \in V$ and all $z \in W$

$$F(q, 0) = f(q),$$

$$F(q, z) \in P \Leftrightarrow z = 0.$$

For its proof, see [**RS18**, Lemma 2.3.5].

2. Vector fields and flows

A vector field on a smooth manifold M is simply a smooth map

$$X : M \to TM,$$

such that for any $p \in M$, $X(p) \in T_pM$. In other words, X is a section for the canonical projection $TM \to M$. We denote by $\Gamma(M)$ the space of vector fields.

Definition 2.1. Let X be a vector field on M and $I \subset \mathbb{R}$ an interval. A smooth map $\gamma : I \to M$ is called an *integral curve* for X if for any $t \in I$ we have

$$\gamma'(t) = X(\gamma(t)). \tag{1}$$

Example 2.2. Let $M = \mathbb{R}$. Then any smooth map $X : \mathbb{R} \to \mathbb{R}$ is a vector field on M. In particular, integral curves for X are the solutions on $I \subset \mathbb{R}$ of the differential equation

$$f' = X(f).$$

Theorem 2.3. *Let $p \in M$ and $X \in \Gamma(M)$. Then we have*

- (i) *There is an open interval $I \subset \mathbb{R}$ containing 0 and an integral curve $\gamma : I \to M$ for X satisfying $\gamma(0) = p$.*
- (ii) *If $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$ are two integral curves for X with $\gamma_1(0) = \gamma_2(0) = p$, then $\gamma_1(t) = \gamma_2(t)$ for every $t \in I_1 \cap I_2$.*

Proof.
- (i) Let $\phi : U \to \mathbb{R}^d$ be a coordinate chart on M, where $U \subset M$ is an open neighborhood of p. Let V its image (which is open) and $\psi = \phi^{-1} : V \to M$. By the proof of Proposition 1.2, we have $T_pM = d_q\psi(\mathbb{R}^d)$, where $q = \phi(p)$. Define $f : V \to \mathbb{R}^d$ by

$$f(x) = d_x\psi^{-1}X(\psi(x)), \;\; x \in V.$$

This map is smooth and hence, by the basic existence and uniqueness theorem for ordinary differential equations in $\mathbb{R}^d$ the equation

$$y' = f(y), \;\; y(0) = q = \phi(p), \tag{2}$$

has a solution $x : I \to \mathbb{R}$ in some open interval $I \subset \mathbb{R}$ containing 0. So the function

$$\gamma := \psi \circ x : I \to U \subset M$$

is an integral curve as desired.
- (ii) The local uniqueness theorem asserts that two solutions $\gamma_i : I_i \to M$ of (i) for $i = 1, 2$ agree on the interval $(-\varepsilon, \varepsilon) \subset I_1 \cap I_2$ for $\varepsilon > 0$ sufficiently small. This follows immediately from the standard uniqueness theorem for the solutions of (i) in [17] and the fact that $x : I \to V$ is a solution of (2) if and only if $\gamma := \psi \circ x : I \to U$ is a solution of (i).
We observe that the set $I := I_1 \cap I_2$ is an open interval containing zero

and hence is connected. Now consider the set $A := \{t \in I | \gamma_1(t) = \gamma_2(t)\}$. This set is nonempty, because $0 \in A$. It is closed, relative to I, because the maps $\gamma_i : I_i \to M$ are continuous. Namely, if $t_i \in I$ is a sequence converging to $t \in I$ then $\gamma_1(t_i) = \gamma_2(t_i)$ for every i and, taking the limit $i \to \infty$, we obtain $\gamma_1(t) = \gamma_2(t)$ and hence $t \in A$. The set A is also open by the local uniqueness theorem. Since I is connected it follows that $A = I$. This proves (ii).

□

The real vector space structure of the tangent space at any point makes it possible to give $\Gamma(M)$ the structure of a real vector space where the addition is defined point by point $(X + Y)(p) = X(p) + Y(p)$ and the scaler multiplication $(\lambda \cdot X)(p) = \lambda X(p)$. In particular, we denote by 0 the zero section. Moreover, the space $\Gamma(M)$ has also the structure of $C^\infty(M)-$module. The multiplication with a function is also defined point by point $(f \cdot X)(p) = f(p)X(p)$.

Now, let $\varphi : M \to N$ be a diffeomorphism, the pullback of a vector field $Y \in \Gamma(N)$ under φ is the vector field on M defined by

$$\varphi^* Y(p) = d_p\varphi^{-1}(Y(\varphi(p))),$$

for $p \in M$. If $X \in \Gamma(M)$, then the pushforward of X under φ is the vector field on N defined by

$$\varphi_* X(q) := d_{\varphi^{-1}(q)}(X(\varphi^{-1}(q))),$$

for $q \in N$.

Definition 2.4. A vector field X on M is called complete if, for each $p \in M$, there is an integral curve $\gamma : \mathbb{R} \to M$ of X with $\gamma(0) = p$.

Lemma 2.5. *Let $M \subset \mathbb{R}^n$ is a compact manifold. Then every vector field on M is complete.*

Proof. See Exercise 0.13. □

Example 2.6. Let $M \subset \mathbb{R}^n$ be an open subset. We know that at each point $p \in M$ the tangent space T_pM can be identified with $\mathbb{R}^n$, and a smooth vector field on M is nothing but a smooth map $X : M \to \mathbb{R}^n$. If we regard tangent vectors to $\mathbb{R}^n$ as directional differentiation operators (see Remark 1.5), the standard basis vectors are the partial derivative operators $\dfrac{\partial}{\partial x_i}$, for $i = 1, \dots, n$. The operator corresponding to X is the partial differential operator

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},$$

where the coefficients $a_1, \dots, a_n \in C^\infty(M)$ are the components of X. In other words, we can think of a smooth vector field on $\mathbb{R}^n$ as a first order partial differential operator with smooth coefficients.

So a vector field X is seen as a linear operator $D_X : C^\infty(M) \to C^\infty(M)$, which associates to function f the function $D_X f : p \mapsto D_{X(p)} f$. This point of view allows as to define an action of $\Gamma(M)$ on $C^\infty(M)$ by letting $X(f) := D_X f$. It also allows us to define a Lie algebra structure on $\Gamma(M)$.

Lie structure. Using this second point of view, we define a *Lie brackets* on $\Gamma(M)$ by

$$[D_X, D_Y] := D_X \circ D_Y - D_Y \circ D_X.$$

None of the operators $D_X \circ D_Y$ and $D_Y \circ D_X$ are vector fields, since they are second order operators, whereas vector fields are first order operators. However, it turns out that their difference $[D_X, D_Y]$ is again a vector field, this is given in the following theorem

THEOREM 2.7. *The Lie bracket $[D_X, D_Y]$ is a smooth vector field on M. That is, there exists a vector field $Z \in \Gamma(M)$ such that $[D_X, D_Y] = D_Z$.*

PROOF. Let $p \in M$, and let (U, ϕ) a coordinate chart around p. Write

$$D_X = \sum_i a_i \frac{\partial}{\partial x_i}, \quad D_Y = \sum_i b_i \frac{\partial}{\partial x_i}.$$

Then, we have for any smooth map f

$$\begin{aligned}
[D_X, D_Y]_p f &= \sum_i a_i \frac{\partial}{\partial x_i}\left(\sum_j b_j \frac{\partial f}{\partial x_j}\right) - \sum_j b_j \frac{\partial}{\partial x_j}\left(\sum_i a_i \frac{\partial f}{\partial x_i}\right) \\
&= \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} + a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{i,j} b_j \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} - a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j} \\
&= \sum_i \left(\sum_j a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial f}{\partial x_i}.
\end{aligned}$$

Thus

$$[D_X, D_Y] = \sum_i \left(\sum_j a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial}{\partial x_i}.$$

□

REMARK 2.8. We can prove the above Theorem by showing that $[D_X, D_Y]$ verifies the Leibniz rule. Let $f, g \in C^\infty(M)$, then we have

$$\begin{aligned}
[D_X, D_Y](fg) &= (D_X \circ D_Y - D_Y \circ D_X)(fg) = D_X(D_Y(fg)) - D_Y(D_X(fg)) \\
&= D_X(D_Y(f)g + fD_Y(g)) - D_Y(D_X(f)g + fD_X(g)) \\
&= D_X(D_Y(f))g + D_Y(f)D_X(g) + D_X(f)D_Y(g) + fD_X(D_Y(g)) \\
&- D_Y(D_X(f))g - D_X(f)D_Y(g) - D_Y(f)D_X(g) - fD_Y(D_X(g)) \\
&= [D_X, D_Y](f)g + f[D_X, D_Y](g).
\end{aligned}$$

REMARK 2.9. One can define the Lie brackets on $\Gamma(M)$ as follows:

$$[X, Y](p) := d_pX(Y(p)) - d_pY(X(p)).$$

Question: Relate the two definitions of the Lie brackets.

Flows of a vector field. Let M as above and $X \in \Gamma(M)$. Let $p \in M$, the *maximal existence interval* of p is the open interval $I(p) := \bigcup I$ where the union runs over all $I \subset \mathbb{R}$ open intervals containing 0 such that there exists an integral curve $\gamma : I \to M$ for X with $\gamma(0) = p$.
By Theorem 2.3, there exists an integral curve $\gamma : I(p) \to M$. The *flow* of X is the map $\phi : \mathcal{D} \to M$ defined by

$$\mathcal{D} := \{(t,p) | p \in M, t \in I(p)\},$$

and $\phi(t,p) := \gamma(t)$, where $\gamma : I(p) \to M$ is the unique integral curve.

THEOREM 2.10. *Let $M \subset \mathbb{R}^n$ be a smooth $r-$manifold and $X \in \Gamma(M)$ be a smooth vector field on M. Let $\phi : \mathcal{D} \to M$ be the flow of X. Then the following holds.*

(*i*) *Let $p \in M$ and $s \in I(p)$. Then*

$$I(\phi(s,p)) = I(p) - s, \tag{3}$$

and, for every $t \in \mathbb{R}$ with $s + t \in I(p)$, we have

$$\phi(s+t,p) = \phi(t, \phi(s,p)). \tag{4}$$

(*ii*) *$\mathcal{D}$ is an open subset of $\mathbb{R} \times M$.*
(*iii*) *The map $\phi : \mathcal{D} \to M$ is smooth.*

PROOF. (*i*) The map $\gamma : I(p) - s \to M$ defined by $\gamma(t) := \phi(s+t,p)$ is a solution of the initial value problem $\gamma'(t) = X(\gamma(t))$ with $\gamma(0) = \phi(s,p)$. Hence $I(p) - s \subset I(\phi(s,p))$ and equation (4) holds for every $t \in \mathbb{R}$ with $s + t \in I(p)$. In particular, with $t = -s$, we have $p = \phi(-s, \phi(s,p))$. Thus we obtain equality in equation (3) by the same argument with the pair (s,p) replaced by $(-s, \phi(s,p))$.

(*ii*) & (*iii*) Let $(t_0, p_0) \in \mathcal{D}$ so that $p_0 \in M$ and $t_0 \in I(p_0)$. Suppose $t_0 \geqslant 0$. Then $K := \{\phi(t,p_0) | 0 \leqslant t \leqslant t_0\}$ is a compact subset of M. (It is the image of the compact interval $[0,t_0]$ under the unique solution $\gamma : I(p_0) \to M$ of equation (1).) Hence, by Lemma 2.11 bellow, there is an $M-$open set $U \subset M$ and an $\varepsilon > 0$ such that

$$K \subset U, \;\; (-\varepsilon, \varepsilon) \times U \subset \mathcal{D},$$

and ϕ is smooth on $(-\varepsilon,\varepsilon) \times U$. Choose N so large that $t_0/N < \varepsilon$. Define $U_0 := U$ and, for $k = 1, \cdots, N$, define the sets $U_k \subset M$ inductively by

$$U_k := \{p \in U | \phi(t_0/N, p) \in U_{k-1}\}.$$

These sets are open in the relative topology of M. We prove by induction on k that $(-\varepsilon, kt_0/N + \varepsilon) \times U_k \subset D$ and ϕ is smooth on $(-\varepsilon, kt_0/N+\varepsilon) \times U_k$. For $k = 0$ this holds by definition of ε and U. If $k \in \{1, \cdots, N\}$ and the assertion holds for $k - 1$ then we have

$$\begin{aligned} p \in U_k &\Rightarrow p \in U, \;\; \phi(t_0/N, p) \in U_{k-1} \\ &\Rightarrow (-\varepsilon, \varepsilon) \subset I(p), \;\; (-\varepsilon, (k-1)t_0/N + \varepsilon) \subset I(\phi(t_0/N, p)) \\ &\Rightarrow (-\varepsilon, kt_0/N + \varepsilon) \subset I(p). \end{aligned}$$

Here the last implication follows from Equation (3). Moreover, for $p \in U_k$ and $t_0/N - \varepsilon < t < kt_0/N + \varepsilon$, we have, by Equation (4), that

$$\phi(t,p) = \phi(t - t_0/N, \phi(t_0/N, p)).$$

Since $\phi(t_0/N, p) \in U_{k-1}$ for $p \in U_k$ the right hand side is a smooth map on the open set $(t_0/N - \varepsilon, kt_0/N + \varepsilon) \times U_k$. Since $U_k \subset U$, ϕ is also a smooth map on $(-\varepsilon, \varepsilon) \times U_k$ and hence on $(-\varepsilon, kt_0/N + \varepsilon) \times U_k$. This completes the induction. With $k = N$ we have found an open neighborhood of (t_0, p_0) contained in $\mathcal{D}$, namely the set $(-\varepsilon, t_0 + \varepsilon) \times U_N$, on which ϕ is smooth. The case $t_0 \leqslant 0$ is treated similarly. This proves (ii) and (iii).

□

LEMMA 2.11. *Let M, X, $\mathcal{D}$ be as in Theorem 2.10 and let $K \subset M$ be a compact set. Then there exists an $M-$open set $U \subset M$ and an $\varepsilon > 0$ such that $K \subset U$, $(-\varepsilon, \varepsilon) \times U \subset D$, and ϕ is smooth on $(-\varepsilon, \varepsilon) \times U$.*

PROOF. see [**RS18**, Lemma 2.4.10] □

PROPOSITION 2.12. *Let $M \subset \mathbb{R}^n$ be a smooth submanifold and $X \in \Gamma(M)$. Then the following are equivalent.*

- (i) *X is complete.*
- (ii) *$I(p) = \mathbb{R}$ for all $p \in M$.*
- (iii) *$\mathcal{D} = \mathbb{R} \times M$.*

PROOF. Clear. □

Let us denote the space of diffeomorphisms of M by

$$\mathrm{Diff}(M) := \{f : M \to M \,|\, f \text{ is a diffeomorphism}\}.$$

This is a group. The group operation is composition and the neutral element is the identity. Now equation (4) asserts that the flow of a complete vector field $X \in \Gamma(M)$ is a group homomorphism

$$\mathbb{R} \to \mathrm{Diff}(M) : t \to \phi^t.$$

This homomorphism is smooth and is characterized by the equation

$$\frac{d}{dt}\phi^t(p) = X(\phi^t(p)), \;\; \phi^0(p) = p,$$

for all $p \in M$.

THEOREM 2.13 (Normal form around a point). *Let $X \in \Gamma(M)$ and $p \in M$ such that $X(p) \neq 0$. Then there exists a local chart (U, ϕ) around p such that if $\phi = (x_1, \cdots, x_d)$ then*

$$X|_U = \frac{\partial}{\partial x_1}.$$

In particular, $\phi^t(x_1, \cdots, x_d) = (t + x_1, \cdots, x_d)$.

PROOF. Let (U, ψ) be any local coordinate chart around p with $\psi(p) = 0$. Since we can compose this with a diffeomorphism of the target, we can assume that

$$X(p) = \frac{\partial}{\partial x_1}(p).$$

Define

$$\chi(y_1, \cdots, y_d) = \phi^{y_1}(\psi^{-1}(0, y_2, \cdots, y_d)).$$

The differential $d_0\chi$ of this map at 0 is bijective. Indeed, one can compute

$$d_0\chi = d_p\left((\phi^t)'|_0\right) \circ d_0\psi^{-1}.$$

Using the equality
$$d_p\left((\phi^t)'|_0\right) = d_{\phi^0(p)=p}X \circ d_p\phi^0$$
and the fact that $\phi^0 = id$, we can see that $d_0\chi$ sends the coordinate e_i to the basis $\dfrac{\partial}{\partial x_i}$. □

3. Parallel transport

Let $M \subset \mathbb{R}^n$ be a submanifold. The Euclidean inner product $\langle\ ,\ \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ induces an inner product
$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}.$$
The set $\{g_p\}_p$ is called the *first fundamental form.*

Note that this inner product induces an orthogonal projection $\Pi_p : \mathbb{R}^n \to T_pM$ for each $p \in M$. The linear map Π_p can be represented by $n \times n$ matrix over $\mathbb{R}$, and it is uniquely determined by the two conditions
$$\Pi_p = \Pi_p^2,$$
and for $v \in \mathbb{R}^n$
$$\Pi_p(v) = v \ \Leftrightarrow\ v \in T_pM.$$

REMARK 3.1. The map $\Pi : M \to \mathbb{R}^{n\times n}$ is smooth.

The differential of Π is the linear map $d_p\Pi : T_pM \to \mathbb{R}^{n\times n}$ which associate to a vector $v = \gamma'(0) \in T_pM$ the matrix
$$d_p\Pi(v) := \frac{d}{dt}|_{t=0}\Pi(\gamma(t)) \in \mathbb{R}^{n\times n}.$$

LEMMA 3.2. *For any $v, w \in T_pM$ we have*
$$d_p\Pi(v)\cdot w = d_p\Pi(w)\cdot v \in T_pM^{\perp}.$$

The collection of symmetric bilinear maps $h_p : (v,w) \mapsto h_p(v,w) := d_p\Pi(v)\cdot w$ is called the *second fundamental form.*

Let $M \subset \mathbb{R}^n$ be a submanifold and $\gamma : I \to M$ a smooth curve. A *vector field along* γ is a section of TM over I, that's a smooth map $X_\gamma : I \to \mathbb{R}^n$ such that $X_\gamma(t) \in T_{\gamma(t)}M$. The derivative $X_\gamma'(t)$ is not in general in the tangent space $T_{\gamma(t)}M$.

DEFINITION 3.3. The covariant derivative of X_γ is the vector field ∇X_γ over γ defined by
$$\nabla X_\gamma := \Pi_p(X_\gamma'(t)) \in T_{\gamma(t)}M.$$

LEMMA 3.4. *Let X_γ as above and $\lambda : I \to \mathbb{R}$ be a smooth function. Then we have*
$$\nabla(\lambda X_\gamma) = \lambda' X_\gamma + \lambda\nabla(X_\gamma).$$

DEFINITION 3.5. Let $I \subset \mathbb{R}$ be an interval and let $\gamma : I \to M$ be a smooth curve. A vector field X_γ along γ is called *parallel* if
$$\nabla X_\gamma(t) = 0.$$
for all $t \in I$.

In other words, X_γ is parallel if and only if $X_\gamma'(t) \perp T_{\gamma(t)}M$.

EXAMPLE 3.6. In particular, γ' is a vector field along γ and $\nabla\gamma'(t) = \Pi_{\gamma'(t)}(\gamma''(t))$. Hence γ' is a parallel vector field along γ if and only if $\gamma''(t) \perp T_{\gamma(t)}M$ for all $t \in I$.

THEOREM 3.7. *Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \to M$ be a smooth curve. Let $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)}M$ be given. Then there is a unique parallel vector field X_γ along γ such that $X_\gamma(t_0) = v_0$.*

DEFINITION 3.8 (Parallel transport). Let $I \subset \mathbb{R}$ be an interval and let $\gamma : I \to M$ be a smooth curve. For $t_0, t \in I$ we define the map

$$\Phi(t, t_0) : T_{\gamma(t_0)}M \longrightarrow T_{\gamma(t)}M$$

by

$$\Phi(t, t_0)(v_0) := X_\gamma(t),$$

where X_γ is the unique parallel vector field along γ satisfying $X_\gamma(t_0) = v_0$. The collection of maps $\Phi(t, t_0)$ for $t, t_0 \in I$ is called parallel transport along γ.

CHAPTER 3

Differential forms

1. Tensors

1.1. Exterior product. Let $k \geqslant 0$, and let V be a finite dimensional vector space over $\mathbb{R}$. The exterior product $\Lambda^k V$ of V is defined by

$$\Lambda^k V = \bigotimes^k V / \sim,$$

where the equivalence relation is given by

$$\otimes_{i=1}^k x_i \sim \otimes_{i=1}^k y_i \Leftrightarrow \otimes_{i=1}^k x_i = sng(\sigma) \otimes_{i=1}^k y_{\sigma(i)}, \text{ for some permutation } \sigma.$$

The class of a tensor $x_1 \otimes \cdots \otimes x_k$ is denoted $x_1 \wedge \cdots \wedge x_k$.
Note that the dimension of $\Lambda^k V$ is given by $\binom{n}{k}$, where $n = \dim(V)$.
There is a canonical alternating $k-$linear map $\wedge_k : V^k \to \Lambda^k V$ given by

$$\wedge_k(x_1, \ldots, x_n) = x_1 \wedge \cdots \wedge x_n,$$

such that for any alternating $k-$linear map $f : V^k \to W$, there is a unique linear map $g : \Lambda^n V \to W$ such that $f = g \circ \wedge_k$.

Lemma 1.1. *Assume that $e_1, \ldots, e_n$ is a basis of V. Then $\{e_{i_1} \wedge \cdots \wedge e_{i_k} | i_1 < \cdots < i_k\}$ is a basis of $\Lambda^k V$. In particular,*

$$\dim \Lambda^k V = \binom{n}{k},$$

where $n = dim(V)$.

Proposition 1.2. *The vectors $v_1, \ldots, v_k$ are linearly independent if and only if $v_1 \wedge \cdots \wedge v_k \neq 0$. Two linearly independent $k-$tuples of vectors $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ span the same $k-$dimensional linear subspace if and only if $v_1 \wedge \cdots \wedge v_k = cw_1 \wedge \cdots \wedge w_k$ for some $c \in \mathbb{R}^n \setminus \{0\}$.*

Corollary 1.3. *The Grassmannian manifold $Gr^k(V)$ can be embedded into the projective space $P(\Lambda^k V)$ by assigning to the $k-$dimensional subspace spanned by the linearly independent vectors $v_1, \cdots, v_k$ the $1-$dimensional linear space spanned by $v_1 \wedge \cdots \wedge v_k$. This embedding is called the* Plücker embedding.

The direct sum $\Lambda^*(V) = \bigoplus_{k=1}^n \Lambda^k V$ is an associative algebra where the product $\wedge$ is defined

$$(v_1 \wedge \cdots \wedge v_k) \wedge (w_1 \wedge \cdots \wedge w_l) = v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_l.$$

$\Lambda^* V$ is called the exterior algebra or Grassmannian algebra of V.

DEFINITION 1.4. A (covariant) $k-$tensor on V is a multilinear map $T : V^k \to \mathbb{R}$. The set of $k-$tensors on V is denoted $T^k(V)$.

Let $\phi_1, \dots, \phi_k \in V^*$. The map

$$\phi_1 \otimes \cdots \otimes \phi_k : (v_1, \dots, v_k) \mapsto \phi_1(v_1) \dots \phi_k(v_k)$$

is a $k-$tensor on V. More generally, if $S \in T^k(V)$ and $T \in T^l(V)$ are tensors, we define the tensor product $S \otimes T \in T^{k+l}(V)$ by

$$S \otimes T(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) = S(v_1, \dots, v_k)T(v_{k+1}, \dots, vk + l).$$

LEMMA 1.5. *Let $\xi_1, \dots, \xi_n$ be a basis of V^*. Then the family $\{\xi_1 i_1 \otimes \cdots \otimes \xi_k\}$, where $i_1, \dots, i_k$ are arbitrary number in $\{1, \dots, n\}$, form a basis of $T^k(V)$. In particular* $\dim T^k(V) = n^k$.

So one sees that $T^k(V) \cong V^* \otimes \cdots \otimes V^*$.

DEFINITION 1.6. A $k-$tensor φ is called alternating if for every $v_1, \dots, v_k \in V$ and any $\sigma \in \Sigma_k$, one have

$$\varphi(v_1, \dots, v_k) = \operatorname{sign}(\sigma)\varphi(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

The space of these is denoted $A^k(V)$. For $\phi_1, \dots, \phi_k \in V^*$, the map

$$\phi_1 \wedge \cdots \wedge \phi_k : (v_1, \dots, v_k) \mapsto \sum_{\sigma \in \Sigma} \phi_1(v_{\sigma(1)}) \cdots \phi_k(v_{\sigma(k)})$$

is alternating $k-$tensor.
So we get as before

$$A^k(V) \cong \Lambda^k V^*.$$

REMARK 1.7. There is a canonical map $\Lambda^k V^* \to (\Lambda^k V)^*$ defined by sending an alternating tensor $\varphi_1 \wedge \cdots \wedge \varphi_k$ to the linear form

$$Alt(\varphi_1 \wedge \cdots \wedge \varphi_k)(x_1 \wedge \cdots \wedge x_k) = \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma)\varphi_1(x_{\sigma(1)}) \cdots \varphi_k(x_{\sigma(k)}),$$

which is an isomorphism.

2. Differential forms

Let $M \subset \mathbb{R}^n$ be a smooth submanifold. Let $f \in C^\infty(M)$. The differential of f is a map $df : TM \to \mathbb{R}$, at each fiber, it gives a linear form $d_p f$ of the vector space $T_p M$. Hence an element of the dual space.

Denote by T^*M the cotangent bundle over M; it is the dual of the tangent bundle. Each fiber of T^*M is isomorphic to $(T_p M)^*$.

DEFINITION 2.1. A differentiable $1-$form on M (also called *covector field*) is a section of the cotangent bundle, that's a smooth map $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ for each $p \in M$. The space of $1-$forms is denoted $\Omega^1(M)$.

As we have seen before, each smooth function f gives rise to a $1-$form df. A differentiable $1-$form is called *exact* if it is equal to df for some $f \in C^\infty(M)$. In particular, if f equals one of the coordinates x_i around a point $p \in M$, we get the $1-$form dx_i. Moreover, any $1-$form ω can be written, locally around p in the form

$$\omega = \sum_{i=1}^{n} a_i dx_i.$$

EXAMPLE 2.2. Let $M = \mathbb{R}^2$ with coordinates x, y, and let $f \in C^\infty(\mathbb{R}^2)$. The differential of x and y equal to dx and dy whose Jacobian matrices at any point are equals to $(1, 0)$ and $(0, 1)$ respectively. Hence the differential of f is given by

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

And any other covector field ω is of the form $adx + bdy$, with $a, b \in C^\infty(\mathbb{R}^2)$, because the cotangent bundle is trivial.

More generally, we have

DEFINITION 2.3. A differentiable $k-$form on M is a section of the exterior product $\Lambda^k T^* M$ of the cotangent bundle, that's a smooth map $\omega : M \to \Lambda^k T^* M$ such that $\omega(p) \in \Lambda^k T_p^* M$ for each $p \in M$. The space of $k-$forms is denoted $\Omega^k(M)$.

On a local coordinate chart $U \subset M$, let $x_1, \ldots, x_d$, then $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is $k-$form, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$. Moreover, any $k-$form on U can be written uniquely in the form

$$w = \sum_I a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where a_I are smooth functions on U.

EXAMPLE 2.4. Let $M = \mathbb{R}^2$ with coordinates x, y. A differential $2-$form ω on M is of the form $adx \wedge dy$, with $a \in C^\infty(\mathbb{R}^2)$, because the cotangent bundle is trivial and its second exterior power is a line bundle. If $X, Y \in T_p M$, then

$$\omega_p(X, Y) = a(p)(x_1 y_2 - x_2 y_1).$$

We denote by $\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$. The wedge product $\wedge$ gives $\Omega^*(M)$ the structure of an algebra over $C^\infty(M)$.

2.1. Interior product. Let ω a differential $k-$form on a manifold M, and X a smooth vector field. We have

DEFINITION 2.5. The interior product of X and ω is the differential $(k-1)-$form $\iota_X \omega$ defined by

$$\iota_X \omega(X_1, \ldots, X_{k-1}) := \omega(X, X_1, \ldots, X_{k-1}).$$

If $k = 0$, we set $\iota_X \omega = 0$.

Interior product ι_X with the vector field X can be thought of as a linear map from $\Omega^*(M)$ into itself. This is a degree -1 map, as it decreases the degree of any form by 1. In general, a linear map $L : \Omega^*(M) \to \Omega^*(M)$ is a degree d map, if $L(\Omega^k(M)) \subset \Omega^{k+d}(M)$ for all k.

2.2. Exterior derivative. Let $M \subset \mathbb{R}^n$ be a submanifold. In a local chart around a point $p \in M$, we can write a $k-$form $\omega \in \Omega^k$ in the form

$$\omega = \sum_{i_1 < \cdots < i_d} a_I dx_{i_1} \wedge \cdots \wedge dx_{i_d},$$

We get then

DEFINITION 2.6. The exterior derivative of the $k-$form ω is the $(k+1)-$form $d\omega$ giving locally by

$$d\omega = \sum_{i_1 < \cdots < i_d} da_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_d}.$$

It is a non trivial result that this definition can be globalized to get a morphism

$$d : \Omega^k(M) \to \Omega^{k+1}(M).$$

DEFINITION 2.7. A $k-$form is called *exact* if it equals $d\alpha$ form some $(k-1)-$form α. It is called *closed* if $d\omega = 0$.

Since $d^2 = 0$ by definition, we deduce

LEMMA 2.8. *An exact form is closed. That's the exterior derivative defines a complex*

$$0 \longrightarrow \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M) \to 0,$$

meaning $Im(d) \subset Ker(d)$. Here $m = dim(M)$.

EXAMPLE 2.9. Reconsider the example 2.2. The $1-$form $\omega = adx + bdy$ is exact if and only if

$$\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}.$$

Indeed, if ω is exact then there is an $f \in C^\infty(\mathbb{R}^2)$ such that $a = \dfrac{\partial f}{\partial x}, b = \dfrac{\partial f}{\partial y}$. Hence

$$\frac{\partial a}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial b}{\partial x}.$$

Conversely, assume that $\dfrac{\partial a}{\partial y} = \dfrac{\partial b}{\partial x}$. Then choose a smooth two variable function f such that $\dfrac{\partial f}{\partial x} = a$. Then $\dfrac{\partial a}{\partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial b}{\partial x}$. So $b - \dfrac{\partial f}{\partial y}$ doesn't depend on x. Choose a function g on y such that $g'(y) = b - \dfrac{\partial f}{\partial y}$. Then one sees that $\omega = \mathrm{d}(f+g)$. Note that this in not true in arbitrary manifold.

Let $f : M \to N$ a smooth map and let $\omega \in \Omega^k(N)$. We define the *pull back* of ω by f to be the $k-$form denoted $f^*\omega$ on M given by

$$f^*\omega(p)(v_1, \cdots, v_k) = \omega(f(p))(d_pf(v_1), \cdots, d_pf(v_k)),$$

for all $v_1, \cdots, v_k \in T_pM$ and all $p \in M$.

CHAPTER 4

Differential manifolds

1. Abstract manifolds

Let T be a topological space. A neighborhood of a point $x \in T$ is a subset $U \subset T$ with the property that it contains an open set containing x. A map $f : T \to W$ which is continuous, bijective and has a continuous inverse is called a homeomorphism.

Definition 1.1. A topological space X is said to be Hausdorff if for every pair of distinct points $x, y \in X$ there exist disjoint neighborhoods of x and y.

Let M be a Hausdorff topological space, and let $m \geqslant 0$ be a fixed non negative integer.

Definition 1.2. An $m-$dimensional smooth atlas of M is a collection $(O_i)_{i\in I}$ of open sets $O_i \subset M$ such that $M = \cup_{i\in I} O_i$, together with a collection $(U_i)_{i\in I}$ of open sets in $\mathbb{R}^m$ and a collection of homeomorphisms, called charts, $\phi_i : O_i \to U_i = \phi_i(U_i)$, with the following property of smooth transition on overlaps:
For each pair $i, j \in I$ the map $\phi_j \circ \phi_i^{-1} : \phi_i(O_i \cap O_j) \to \phi_j(O_i \cap O_j)$ is smooth.
Two smooth atlases are compatible if their union is again an atlas.

If $\mathscr{A}$ is an atlas, then so is the collection $\bar{\mathscr{A}}$ of all charts compatible with each member of $\mathscr{A}$. The atlas $\bar{\mathscr{A}}$ is obviously maximal. In other words, every atlas extends uniquely to a maximal atlas.

Example 1.3. Assume that $W \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets, that M is a subset of the product $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$, and $f : W \to V$ is a map whose graph is a subset of M, i.e.

$$graph(f) := \{(x, y)|x \in W,\ y = f(x)\} \subset M.$$

Let $U = (W \cap V) \smallsetminus graph(f)$ and let $\phi(x, y) = x$ be the projection of U onto W. Then the pair (ϕ, U) is a chart on M. The inverse map is given by $\phi^{-1}(x) = (x, f(x))$.

Definition 1.4. An abstract manifold (or just a manifold) of dimension m, is a Hausdorff topological space M, equipped with a maximal $m-$dimensional smooth atlas.

Remark 1.5. If in the definition, we don't assume that M is a topological space, and that O_i are just subset, then there is a unique topology on M making O_i open and ϕ_i homeomorphisms.

Example 1.6. The stereographic projections endows the spheres S^n with an atlas given as follows; let $U_\pm$ be the subsets of S^n obtained by removing $(\mp 1, 0, \cdots, 0)$.

Stereographic projection defines bijections $\phi_{\pm} : U_{\pm} \to \mathbb{R}^n$ given by

$$\phi(x_0, \cdots, x_n) = (\frac{x_0}{1 \pm x_0}, \cdots, \frac{x_n}{1 \pm x_0}).$$

One can check (exercise!) that

$$\phi_- \circ \phi_+^{-1}(x) = \frac{x}{\|x\|^2},$$

hence it is smooth.

Definition 1.7. Let $f : M \to N$ be a map between abstract manifolds. Then f is called smooth if for each $p \in M$ there exists a chart $\phi : O \to U$ around p, and a chart $\psi : O' \to V$ around $f(p)$, such that $f(O) \subset O'$ and such that the coordinate expression $\psi \circ f \circ \phi^{-1}$ is smooth. A bijective map $f : M \to N$, is called a diffeomorphism if f and f^{-1} are both smooth.

Notice that a smooth map $f : M \to N$ is continuous. This follows immediately from the definition above, by writing $f = \psi^{-1} \circ (\psi \circ f \circ \phi^{-1}) \circ \phi$ in a neighborhood of each point. Also, every chart $\phi : O \to U$ is a smooth diffeomorphism. And every diffeomorphism ϕ of a non-empty open subset $V \subset \mathbb{R}^m$ onto an open subset in M is a chart on M.

Now, an atlas of an abstract manifold is said to be *countable* if the set of charts in the atlas is countable

Lemma 1.8. *Let M be an abstract manifold. Then M has a countable atlas if and only if there exists a countable base for the topology in M. In particular, a submanifold of $\mathbb{R}^n$ has a countable atlas.*

Proof. Assume M has a countable atlas. Let $\phi : O \to U \subset \mathbb{R}^m$ be a chart, then there is a countable base for the topology of U. This induces a countable base for the topology on O, because ϕ is a homeomorphism. The collection of these bases for all the charts in the atlas gives a base for the topology of M.
Conversely, suppose that there exists a countable base $(V_k)_k$ of the topology of M. For each k, choose a chart $\phi : O \to U$, such that $V_k \subset O$ (if it exists!). This gives a countable atlas $\mathscr{A}$ in M. Indeed, it is clearly countable. Let now $x \in M$. Then there exists a chart $\psi : O' \to U'$ such that $x \in O'$, and there exists k such that $x \in V_k \subset O'$. Hence there exists a chart $\phi : O \to U$ in $\mathcal{A}$ such that $x \in O$. So $\mathscr{A}$ is an atlas. $\square$

Theorem 1.9 (Whitney theorem). *Let M be an abstract smooth manifold of dimension d, and assume that there exists a countable atlas for M. Then there exists a diffeomorphism of M onto a submanifold in $\mathbb{R}^{2d}$.*

Example 1.10 (Projective space). Let V be a finite dimensional vector space, The *projective space* of V is the set of lines (through the origin) in V . In other words,

$$P(V) = \{l \subset V | l \text{ is a 1-dimensional } \mathbb{R}\text{-linear subspace}\}.$$

It can be seen a $P(V) = V^*/\sim$, where the equivalence relation is given by

$$u \sim v \Leftrightarrow u = \lambda v \text{ for some } \lambda \in \mathbb{R}.$$

If $V = \mathbb{R}^n$, then $P(V)$ is denoted simply $\mathbb{P}^n$. The projective space $\mathbb{P}^n$ can be given the structure of an abstract $n-$dimensional manifold as follows. Let $\pi : x \to [x]$

denote the natural map $\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, and let $S^n \subset \mathbb{R}^{n+1}$ denote the unit sphere. A set $U \subset \mathbb{P}^n$ is declared to be open if and only if its preimage $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1}$ (or equivalently, if $\pi^{-1}(U) \cap S^n$ is open in S^n). This makes $\mathbb{P}^n$ a Hausdorff topological space (it is actually the quotient topology, that's a map $f : \mathbb{P}^n \to Y$ is continuous if and only if $f \circ \pi$ is continuous).
For $i = 1, \ldots, n+1$, let $O_i = \{[x] \in \mathbb{P}^n | x_i \neq 0\}$. They are clearly open and cover $\mathbb{P}^n$. Define the maps $\phi_i : O_i \to \mathbb{R}^n$ given by

$$\phi_i([x_1 : \cdots : x_{n+1}]) = (x_1/x_i, \cdots, x_{n+1}/x_i),$$

where the i^{th} component is omitted. Clearly ϕ_i is a homeomorphism from O_i to $\mathbb{R}^n$. Moreover, the collection $\{(O_i, \phi_i)\}$ is a smooth atlas. Indeed, we have

$$\phi_i^{-1} : \mathbb{R}^n \to O_i, \ (x_1, \cdots, x_n) \mapsto [x_1 : \cdots : 1 : \cdots : x_n],$$

where 1 is in i^{th} position. Hence $\phi_j \circ \phi_i^{-1} : \mathbb{R}^n \setminus \{x_j \neq 0\} \to \mathbb{R}^n$

$$\phi_j \circ \phi_i^{-1}(x_1, \cdots, x_n) = (x_1/x_j, \ldots, 1/x_j, \cdots, x_n/x_j),$$

which is smooth.

2. Tangent space

Let M be an $m-$dimensional manifold. A *curve* γ on M is a smooth map $\gamma : I \to M$, where $I \subset \mathbb{R}$ is open. This means (see Definition 1.7) that $\phi \circ \gamma$ is smooth for all chart ϕ on M. The expression $\phi \circ \gamma$ is called *coordinate expression* of γ with respect to ϕ. If $p \in M$ is a point, a parametrized curve on M through p, is a parametrized curve on M together with a point $t_0 \in I$ for which $p = \gamma(t_0)$.

Let $\gamma_i : I_i M$, for $i = 1, 2$, be parametrized curves on M with $p = \gamma(t_1) = \gamma(t_2)$, and let ϕ be a chart around p. We say that γ_1 and γ_2 are tangential at p, if the coordinate expressions satisfy

$$(\phi \circ \gamma)'(t_1) = (\phi \circ \gamma)'(t_2).$$

Lemma 2.1. *Being tangential at p is an equivalence relation on curves through p. It is independent of the chosen chart (ϕ, O).*

Proof. The first part is easy.
If $\tilde{\phi}$ is another chart then the coordinate expressions are related by

$$\tilde{\phi} \circ \gamma = \tilde{\phi} \circ \phi^{-1} \circ (\phi \circ \gamma)$$

on the overlap $O \circ \tilde{O}$. The chain rule implies

$$(\tilde{\phi} \circ \gamma_i)'(t_i) = D(\tilde{\phi} \circ \gamma_i)(x)(\phi \circ \gamma_i)'(t_i),$$

for each of the curves. This implies the result. □

Denote the tangential equivalence relation at p by $\sim_p$.

Definition 2.2. The tangent space T_pM is the set of $\sim_p$ −classes of parametrized curves on M through p.

Theorem 2.3. *Let (ϕ, U) be a chart on M with $\phi(p) = q$. For each element $v \in \mathbb{R}^m$ let $\gamma_v(t) = \phi^{-1}(q + tv)$ for t close to 0. The map*

$$\Gamma : v \mapsto \gamma_v'(0)$$

is a bijection of $\mathbb{R}^m$ onto T_pM. The inverse map is given by

$$\gamma'(0) \mapsto (\phi \circ \gamma)'(0),$$

for each curve γ on M with $\gamma(0) = p$.

3. Orientability

Let V be a finite dimensional vector space. Two ordered bases $(u_1, ..., u_n)$ and $(v_1, ..., v_n)$ are said to be equally oriented if the transition matrix S, whose columns are the coordinates of the vectors $(u_1, ..., u_n)$ with respect to the basis $(v_1, ..., v_n)$, has positive determinant. Being equally oriented is an equivalence relation among bases, for which there are precisely two equivalence classes. The space V is said to be oriented if a specific class has been chosen, this class is then called the orientation of V, and its member bases are called positive. The Euclidean spaces $\mathbb{R}^n$ are usually oriented by the class containing the standard basis $(e_1, ..., e_n)$. For the null space $V = \{0\}$ we introduce the convention that an orientation is a choice between the signs $+$ and $-$.

Let now a (ϕ, U) be a chart on M with $\phi(p) = q$, we obtain a basis for TpM by taking the pre-images (ei) of each of the standard basis vectors e1, . . . , em for Rm (in that order), that is, the basis vectors will be the equivalence classes of the curves t 7 (x0+tei). This basis for TpM is called the standard basis with respect to . The compatibility condition between charts (O_i, ϕ_i) and (O_j, ϕ_j) on a set M is that the map $\phi_j \circ \phi_j^{-1} : \phi_j(O_i \cap O_j) \to \phi_i(O_i \cap O_j)$ is a diffeomorphism. In particular, the Jacobian matrix $J(\phi_j \circ \phi_j^{-1})$ of the transition map is invertible, and hence has nonzero determinant. If the determinant is > 0 everywhere, then we say (O_i, ϕ_i), (O_j, ϕ_j) are oriented-compatible. An oriented atlas on M is an atlas such that any two of its charts are oriented-compatible; a maximal oriented atlas is one that contains every chart that is oriented-compatible with all charts in this atlas.

Definition 3.1. A manifold is called orientable if it admits an oriented atlas.

If an orientation has been chosen we say that M is an oriented manifold. The notion of an orientation on a manifold is crucial, since integration of differential forms over manifolds is only defined if the manifold is oriented.

Let (ϕ, O_i) be a chart on an abstract manifold M, then the tangent space is equipped with the standard basis with respect to ϕ. For each $p \in O_i$ we say that the orientation of T_pM, for which the standard basis is positive, is the orientation induced by ϕ.

Example 3.2. The spheres S^n are orientable. To see this, consider the atlas with the two charts (U_+, ϕ_+) and (U, ϕ_-), given by stereographic projections. (see Example 1.6). Here

$$\phi_- : (U_+ \cap U) = \phi_+(U_+ \cap U) = \mathbb{R}^n \setminus \{0\}.$$

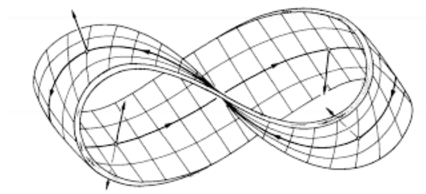
The entries of the Jacobian matrix of $\phi_- \circ \phi_+^{-1}$ at x are given by

$$\frac{\partial}{\partial x_j} \frac{x_i}{\|x\|^2} = \frac{1}{\|x\|^2}\delta_{ij} - \frac{2x_i x_j}{\|x\|^4}. \tag{5}$$

Its determinant is $-\|x\|^{-2n}$ (exercise).

Example 3.3. The real projective space $\mathbb{P}^n$ is orientable if and only if n is odd or $n = 0$ (see exercise 0.31).

EXAMPLE 3.4. a Möbius strip is a surface with only one side.



It is maybe the simplest example of a non-orientable manifold because it is impossible to make a consistent choice of orientations, because the band is one-sided. Choosing an orientation in one point forces it by continuity to be given in neighboring points, and eventually we are forced to the opposite choice in the initial point.

CHAPTER 5

Exercises

Exercise 0.1. For which values of $(a, b) \in \mathbb{R}^2$, the function $f : \mathbb{R}^2 \to \mathbb{R}^2; (x, y) \mapsto (x + a \sin y, y + b \sin x)$ is a local diffeomorphism at any point ? Show that in this case, f is a global diffeomorphism.

Exercise 0.2. Let $B(0, 1)$ be the open ball in $\mathbb{R}^n$ endowed with the Euclidean norm. Show that $f : B(0, 1) \to \mathbb{R}^n$ given by $x \mapsto \dfrac{x}{1 - \|x\|^2}$ is a smooth diffeomorphism.

Exercise 0.3. Let I_1 and I_2 be two intervals in $\mathbb{R}$ and let $f_1 : I_1 \to \mathbb{R}$ and $g : I_2 \to \mathbb{R}$ be two C^1-functions. Define $F : I_1 \times I_2 \to \mathbb{R}^2$ by

$$F(x, y) = (x + y, f(x) + g(y)).$$

(1) Give a sufficient condition for which F is invertible in a neighborhood of $(x_0, y_0) \in I_1 \times I_2$.
(2) Assume for every $(x, y) \in I_1 \times I_2$, $f'(x) \neq g'(y)$. Show that F is a C^1-diffeomorphism from $I_1 \times I_2$ to $F(I_1 \times I_2)$. *Hint: use the mean value theorem.*

Exercise 0.4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-function such that $x \mapsto f(x) - x$ is a $k-$Lipschitz continuous function with $0 < k < 1$. Show that f is a C^1-diffeomorphism from $\mathbb{R}^n$ to itself.

Exercise 0.5. • Show that a subset $M \subset \mathbb{R}^n$ is a $0-$dimensional submanifold if and only if M is discrete, i.e. for every $p \in M$, there is an open set $U \subset \mathbb{R}^n$ such that $U \cap M = \{p\}$.
• Show that a subset $M \subset \mathbb{R}^n$ is an $n-$dimensional submanifold if and only if M is open.

Exercise 0.6. If $M_i \subset \mathbb{R}^{n_i}$ is an d_i-manifold for $i = 1, 2$, show that $M_1 \times M_2$ is an $(d_1 + d_2)$-dimensional submanifold of $\mathbb{R}^{n_1+n_2}$. Prove by induction that the $n-$torus T^n is a smooth submanifold of $\mathbb{R}^{2n}$.
Note: The question is easier then prove that it is a submanifold of $\mathbb{R}^{n+1}$. $T^1 := S^1$, $T^2 = \{(x, y, z) \in \mathbb{R}^3 | (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\}$, $T^{n+1} = T^n \times S^1$.

Exercise 0.7. Let n and m be integers with $0 \leqslant m \leqslant n$. Let $M \subset \mathbb{R}^n$ be a non-empty set and $p \in M$. Then show that the following are equivalent.

(i) There exists an $M-$open neighborhood $U \subset M$ of p and a diffeomorphism

$$\phi : U \to V$$

onto an open set $V \subset \mathbb{R}^m$.

(*ii*) There exist open sets $U, V \subset \mathbb{R}^n$ and a diffeomorphism $\phi : U \to V$ such that $p \in U$ and
$$\phi(U \cap M) = V \cap (\mathbb{R}^m \times \{0\})$$
(*iii*) There exists an open set $U \subset \mathbb{R}^n$ and a smooth map $f : U \to \mathbb{R}^{n-m}$ such that $p \in U$, the differential $d_x f : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is surjective for every $x \in U \cap M$, and
$$U \cap M = f^{-1}(0) = \{q \in U | f(q) = 0\}.$$

Exercise 0.8. Consider the general linear group
$$GL(n, \mathbb{R}) = \{g \in \mathbb{R}^{n\times n} | det(g) \neq 0\}.$$
Prove that the derivative of the function $f = det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is given by
$$d_g f(v) = det(g) Tr(g^{-1} v),$$
for every $g \in GL(n, \mathbb{R})$ and every $v \in \mathbb{R}^{n\times n}$. Deduce that the special linear group
$$SL(n, \mathbb{R}) := \{g \in GL(n, \mathbb{R}) | det(g) = 1\}$$
is a smooth submanifold of $\mathbb{R}^{n\times n}$.

Exercise 0.9. Prove that the tangent space of $SL(n, \mathbb{R})$ at the identity matrix is the space
$$\mathfrak{sl}(n, \mathbb{R}) = T_{id} SL(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} | Tr(A) = 0\}.$$
of traceless matrices.

Exercise 0.10. Consider the space $\mathcal{M}(n, \mathbb{R})$ of square matrices of size n and $\mathcal{S}ym(n, \mathbb{R})$ of symmetric matrices of size n.

(1) Prove that the derivative of the function
$$f : \mathcal{M}(n, \mathbb{R}) \to \mathcal{S}ym(n, \mathbb{R}), \;\; A \to A \times {}^t A$$
($\times$: matrix multiplication, ^{t} : matrix transpose) is given by
$$d_X f(A) = X \times {}^t A + A \times {}^t X,$$
for every $X, A \in \mathcal{M}(n, \mathbb{R})$.

(2) Deduce that the orthogonal group
$$\mathrm{O}(n, \mathbb{R}) := \{A \in \mathcal{M}(n, \mathbb{R}) | A^{-1} = {}^t A\}$$
is a smooth submanifold of $\mathcal{M}(n, \mathbb{R})$.

(3) Prove that
$$T_{id}\mathrm{O}(n, \mathbb{R}) = \{A \in \mathcal{M}(n, \mathbb{R}) \mid {}^t A = -A\}.$$

(4) Deduce the dimension of $\mathrm{O}(n, \mathbb{R})$.

Exercise 0.11. Prove that the set $M = \{(x, y) \in \mathbb{R}^2 | xy = 0\}$ is not a submanifold of $\mathbb{R}^2$.
Hint: If $U \subset \mathbb{R}^2$ is a neighborhood of the origin and $f : U \to \mathbb{R}$ is a smooth map such that $U \cap M = f^{-1}(0)$ then $d_0 f = 0$.

Exercise 0.12. Prove that the map $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(t) := (t^2, t^3)$ is proper and injective, but is not an embedding. The image of f is the set $f(\mathbb{R}) = \{(x, y) \in \mathbb{R}^2 | x^3 = y^2\}$. Show that it is not a submanifold.

EXERCISE 0.13. Let $M = U \subset \mathbb{R}^n$ open. Determine the space $\Gamma(M)$.

EXERCISE 0.14. Let $M \subset \mathbb{R}^n$ is a compact manifold. Then every vector field on M is complete.

EXERCISE 0.15. Let $M \subset \mathbb{R}^n$ be a smooth manifold. A vector field X on M is said to have *compact support* if there exists a compact subset $K \subset M$ such that $X(p) = 0$ for every $p \in M \setminus K$. Prove that every vector field with compact support is complete.

EXERCISE 0.16. Let φ, ψ two diffeomorphisms. Prove that

$$(\varphi \circ \psi)^* = \psi^* \circ \varphi^*, \quad (\varphi \circ \psi)_* = \varphi_* \circ \psi_*.$$

EXERCISE 0.17. Prove that the Lie brackets verifies the following two derivation properties:

$$[fX, Y] = f[X, Y] - D_Y f X,$$
$$[X, fY] = f[X, Y] + D_X f Y,$$

for any $X, Y \in \Gamma(M)$ and $f \in C^\infty(M)$.

EXERCISE 0.18. Prove that the flow of $\varphi_* X$ is conjugated to the flow of X, and the flow of $\varphi^* Y$ is conjugated to the flow of Y. More precisely:

$$\phi^t_{\varphi_* X} = \varphi \circ \phi^t_X \circ \varphi^{-1}, \quad \phi^t_{\varphi^* Y} = \varphi^{-1} \circ \phi^t_Y \circ \varphi.$$

EXERCISE 0.19. Prove that if $f : M \to N$ is an embedding then $df : TM \to TN$ is also an embedding.

EXERCISE 0.20. Let $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ be the two coordinate vector fields on $\mathbb{R}^2$ determined by the identity mapping. Describe the vector fields

$$X(x, y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$X(x, y) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Compute their Lie bracket, and determine the flows generated by them.

EXERCISE 0.21. Prove that

$$(\phi^t)^* X = X, \quad (\phi^t)_* X = X,$$

for any $(t, p) \in \mathcal{D}$ and any complete $X \in \Gamma(M)$.

EXERCISE 0.22. Let $U \subset \mathbb{R}^n$ be an open subset. Show that a vector field X over $\gamma : I \to U$ is parallel if and only if it is constant.

EXERCISE 0.23. Let $M \subset \mathbb{R}^n$ be a submanifold. Define the following brackets on $\Gamma(M)$: For $X, Y \in \Gamma(M)$, we set

$$[X, Y] = dX \circ Y - dY \circ X.$$

(1) Prove that this brackets gives a well defined bilinear map on $\Gamma(M)$, that's $[X, Y] \in \Gamma(M)$.
(2) Show that it is in fact a Lie brackets.
(3) Prove that it is actually the same one giving in the course.

EXERCISE 0.24. Let $M = \mathbb{R}^2$ and let $\omega = adx + bdy \in \Omega^1(\mathbb{R}^2)$ with $a, b \in C^\infty(\mathbb{R}^2)$. Show that ω is exact if and only if it is closed, that is, if and only if

$$\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}$$

.

EXERCISE 0.25. Let θ be the argument on $M := \mathbb{R}^2 \setminus \{0\}$. Note that it is not a function on M.

(i) Give the expression of θ.
(ii) Calculate $d\theta$.
(iii) Show that the $1-$form $d\theta$ on M is closed but not exact, even thought it verifies the condition of Exercise 0.24

EXERCISE 0.26. Show that a function f is a closed differential $0-$form if and only if it is locally constant.

EXERCISE 0.27. Show that for any two vector fields X an Y, we have $\iota_X \circ \iota_Y = -\iota_Y \circ \iota_X$.

EXERCISE 0.28. Show that every diffeomorphism ϕ of a non-empty open subset $V \subset \mathbb{R}^m$ onto an open subset in M is a chart on M.

EXERCISE 0.29. Calculate the determinant of the matrix with entries given in equation (3).

EXERCISE 0.30. Let $A \in \mathrm{Mat}_{\mathbb{C}}(n)$ be a complex $n \times n-$matrix, and $A_{\mathbb{R}} \in \mathrm{Mat}_{\mathbb{R}}(2n)$ the same matrix regarded as a real-linear transformation of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Show that

$$det_{\mathbb{R}}(A_{\mathbb{R}}) = |det_{\mathbb{C}}(A)|^2.$$

Hint: You may start with the case $n = 1$ and then consider the case that A is upper triangular.

EXERCISE 0.31. Show that $\mathbb{P}^n$ is orientable if and only if n is odd or $n = 0$. More generally, $Gr(k, n)$ is orientable if and only if n is even or $n = 1$.

CHAPTER 6

Solutions

Exercise 0.1. We have $\det J_{(x,y)}f = 1 - ab\cos x\cos y$. So f is a local diffeomorphism iff $|ab| < 1$. Now we show that f is injective. let $(x,y),(x',y') \in \mathbb{R}^2$ such that $f(x,y) = f(x',y')$. Then $x - x' = a(\sin y' - \sin y)$ and $y - y' = b(\sin x' - \sin x)$. Assuming $x \neq x'$ and $y \neq y'$, we get $\left|\dfrac{\sin x - \sin x'}{x - x'}\dfrac{\sin y - \sin y'}{y - y'}\right| = \left|\dfrac{1}{ab}\right| > 1$. However, assuming for example $\dfrac{\sin x}{x} \leqslant \dfrac{\sin x'}{x'}$, this implies $\dfrac{\sin x - \sin x'}{x - x'} \leqslant \dfrac{\sin x'}{x'} \leqslant 1$. So $\dfrac{\sin x - \sin x'}{x - x'}\dfrac{\sin y - \sin y'}{y - y'} \leqslant 1$ a contradiction. So f is injective.

Exercise 0.2. Apply the inverse function theorem. Then show that f is bijective. To show that it is injective take the norm on $f(x) = f(y)$. For surjectivity, note that f preserves the lines in $\mathbb{R}^n$.

Exercise 0.3.

Exercise 0.4. Since f is differentiable, we have

$$g(x+h) - g(x) = d_x g(h) + \circ(h) = d_x f(h) - h + \circ(h),$$

using the Lipschitz condition, we get

$$\|d_x f(h) - h\| \leqslant (1+k)\|h\|,$$

for any h, so $d_x f$ is invertible. Do f is a local C^1-diffeomorphism. It remains to show that f is bijective. If $f(x) = f(y)$ then $\|g(x) - g(y)\| = \|x - y\| \leqslant k\|x - y\|$ so $x = y$. Let $y \in \mathbb{R}^n$, we want to show that there exists $x \in \mathbb{R}^n$ such that $f(x) = y = g(x) + x$. This is equivalent to $y - g(x) = x$. So let $F_y : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto y - g(x)$. We see that x is then a fixed point for F_y,. But since F_y is $k-$Lipschitz continuous for $0 < k < 1$, we know that F_y has a fixed point and so f is surjective.

Exercise 0.5.
- Trivial.
- By applying the definition, we deduce that for every $p \in M$ there is open sets $U, V \subset \mathbb{R}^n$ s.t. $p \in U$ and $0 \in V$, and a diffeomorphism $\phi : U \to V$ such that $\phi(U \cap M) = V$. In particular, $U \cap M$ is open and hence M is open too.

Exercise 0.6. Easy.

Exercise 0.7.

Exercise 0.8.

Exercise 0.9.

Exercise 0.10.

EXERCISE 0.11. Using the hint, $\dfrac{\partial f}{\partial x}(x,0) = 0$ ...

EXERCISE 0.12. Let $U \subset \mathbb{R}^2$ be a neighborhood of 0 and let $g : U \to \mathbb{R}$ be smooth s.t. $U \cap f(\mathbb{R}) = g^{-1}(0)$. So $g \circ f = 0$ for any t s.t $f(t) \in U$. Let $a = \dfrac{\partial g}{\partial x}(0,0)$, $b = \dfrac{\partial g}{\partial y}(0,0)$. Then we have for any t as before

$$2at + 3bt^2 = 0.$$

Hence $a = b = 0$.

EXERCISE 0.13. The tangent space T_pU is $\mathbb{R}^n$ for any $p \in U$, since the derivative of $t \mapsto p + tv$ at $t = 0$ is v. Moreover $TU = U \times \mathbb{R}^n$. This implies that a vector field is nothing but a smooth map $U \to \mathbb{R}^n$.

EXERCISE 0.14.

EXERCISE 0.15.

EXERCISE 0.16.

EXERCISE 0.17.

EXERCISE 0.18.

EXERCISE 0.19.

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