TD3

Exercice 01 :

Résolvez l'équation de conduction thermique unidimensionnelle $\partial u/\partial t=2(\partial^2 u/\partial x^2)$; $0\leq x \leq 1$ avec :

Condition initiale : $u(x,0) = x(2 - x)$ et conditions aux limites : $u(0, t) = 0$ et $u(1, t) = 1$

Utilisez un schéma explicite pour trouver la valeur $u(x,t)$ jusqu'à $t = 0.02$, avec $\Delta x = 0.2$ et $\Delta t = 0.005$

Solution :

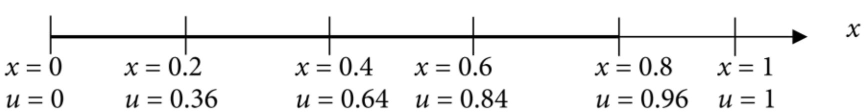
Since $\Delta x = 0.2$ for $0 \leq x \leq 1$, hence our node points are as follows

$$x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8, x_5 = 1$$

Let $u_{i,j} = u(x_i, t_j)$. The initial condition $u(x,0) = x(2-x)$ at $t_0 = 0$ provides

$$u_{0,0} = u(x_0,t_0) = u(0,0) = 0(2-0) = 0$$
$$u_{1,0} = u(x_1,t_0) = u(0.2,0) = 0.2(2-0.2) = 0.36$$
$$u_{2,0} = u(x_2,t_0) = u(0.4,0) = 0.4(2-0.4) = 0.64$$
$$u_{3,0} = u(x_3,t_0) = u(0.6,0) = 0.6(2-0.6) = 0.84$$
$$u_{4,0} = u(x_4,t_0) = u(0.8,0) = 0.8(2-0.8) = 0.96$$
$$u_{5,0} = u(x_5,t_0) = u(1,0) = 1(2-1) = 1 \qquad (16.10)$$

For $t = 0$



Similarly, boundary conditions are $u(0,t) = 0$ and $u(1,t) = 1$, so

$$u_{0,j} = u(x_0,t_j) = u(0, t) = 0$$
$$u_{5,j} = u(x_5,t_j) = u(1, t) = 1; \quad \text{for } \forall\, j = 0,1,2,3,\cdots \qquad (16.11)$$

In the table form, initial values (16.10) and boundary values (16.11) are as follows

$i\ (x)$ / $j\ (t)$	0(0)	1(0.2)	2(0.4)	3(0.6)	4(0.8)	5(1)
0(0)	0	0.36	0.64	0.84	0.96	1
1(0.005)	0					1
2(0.01)	0					1
3(0.015)	0					1
4(0.02)	0					1

After utilizing the initial and boundary conditions, we will now use the explicit method to find the values of $u(x, t)$ at node points. With $\Delta x = 0.2$, $\Delta t = 0.005$ and $c = 2$, our r is given by

$$r = \frac{c\Delta t}{\Delta x^2} = \frac{ck}{h^2} = \frac{1(0.005)}{(0.2)^2} = 0.25$$

Bender–Schmidt Explicit Scheme (16.3) is given by

$$u_{i,j+1} = r\,u_{i-1,j} + (1-2r)u_{i,j} + r\,u_{i+1,j}$$

For $r = 0.25$, we have

$$u_{i,j+1} = 0.25\,u_{i-1,j} + 0.5\,u_{i,j} + 0.25\,u_{i+1,j}$$

Using $j = 0$ in the above formula, we have

$$u_{i,1} = 0.25\,u_{i-1,0} + 0.5\,u_{i,0} + 0.25\,u_{i+1,0}$$

Computing the values for $i = 1, 2, 3$ and 4, we get

$$u_{1,1} = 0.25\,u_{0,0} + 0.5\,u_{1,0} + 0.25\,u_{2,0} = 0.25(0) + 0.5(0.36) + 0.25(0.64) = 0.34$$
$$u_{2,1} = 0.25\,u_{1,0} + 0.5\,u_{2,0} + 0.25\,u_{3,0} = 0.25(0.36) + 0.5(0.64) + 0.25(0.84) = 0.62$$
$$u_{3,1} = 0.25\,u_{2,0} + 0.5\,u_{3,0} + 0.25\,u_{4,0} = 0.25(0.64) + 0.5(0.84) + 0.25(0.96) = 0.82$$
$$u_{4,1} = 0.25\,u_{3,0} + 0.5\,u_{4,0} + 0.25\,u_{5,0} = 0.25(0.84) + 0.5(0.96) + 0.25(0.1) = 0.94$$

These values give the second row of the table.

$j\,(t)$ \ $i\,(x)$	0(0)	1(0.2)	2(0.4)	3(0.6)	4(0.8)	5(1)
0(0)	0	0.36	0.64	0.84	0.96	1
1(0.005)	0	0.34	0.62	0.82	0.94	1
2(0.01)	0					1
3(0.015)	0					1
4(0.02)	0					1

Proceeding in a similar manner, for $j = 1, 2, 3, 4$ and 5, we will get different rows of the table as follows (up to six decimal digits):

$j(t)$ \ $i(x)$	0(0)	1(0.2)	2(0.4)	3(0.6)	4(0.8)	5(1)
0(0)	0	0.36	0.64	0.84	0.96	1
1(0.005)	0	0.34	0.62	0.82	0.94	1
2(0.01)	0	0.325	0.60	0.80	0.925	1
3(0.015)	0	0.3125	0.58125	0.78125	0.9125	1
4(0.02)	0	0.301563	0.564063	0.764063	0.901563	1

Exercice 02 :

Utilisez un schéma explicite pour calculer la distribution de la température dans une tige isolée uniforme de longueur 1 m avec constante de diffusivité du matériau de la tige est donnée 1. Les deux extrémités de la tige sont maintenues à température nulle et la distribution initiale de la température dans la tige est donnée par la fonction u(x,0) = sin(πx). Prenez Δx =1/4, Δt=1/16; résoudre jusqu'à t = 1/8.

Solution :

The diffusivity constant of the material of the rod is 1, i.e. $c = 1$. So, the temperature distribution is given by following heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \qquad 0 \le x \le 1$$

Both ends of the rod are kept at zero temperature; therefore boundary conditions are given by

$$u(0, t) = u(1, t) = 0$$

Also, initial temperature distribution gives following initial condition

$$u(x,0) = \sin(\pi x)$$

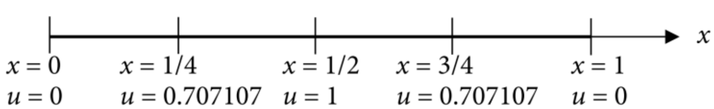
The mathematical model for the given physical problem is complete.

Now, we will compute the temperature distribution at various nodes using explicit scheme

With $\Delta x = 1/4$, $t = 0$, the initial condition $u(x, 0) = \sin(\pi x)$ gives

$$u_{1,0} = u(x_1, t_0) = u(1/4, 0) = \sin(\pi/4) = 1/\sqrt{2} = 0.707107$$
$$u_{2,0} = u(x_2, t_0) = u(1/2, 0) = \sin(\pi/2) = 1$$
$$u_{3,0} = u(x_3, t_0) = u(3/4, 0) = \sin(3\pi/4) = 1/\sqrt{2} = 0.707107 \qquad (16.12)$$

For $t = 0$



The boundary conditions $u(0, t) = u(1, t) = 0$ provide

$$u_{0,j} = u(x_0, t_j) = u(0, t) = 0$$
$$u_{4,j} = u(x_4, t_j) = u(1, t) = 0; \qquad \text{for } \forall\, j = 0, 1, 2, 3, \cdots \tag{16.13}$$

In the table form, (16.12) and (16.13) are as follows

$j\ (t)$ \ $i\ (x)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0				0
2(1/8)	0				0

With $\Delta x = 1/4$, $\Delta t = 1/16$ and $c = 1$, the value of constant r is given by

$$r = \frac{c\Delta t}{\Delta x^2} = \frac{1(1/16)}{(1/4)^2} = 1$$

Explicit scheme (16.3) for $r = 1$ is as follows

$$u_{i,j+1} = u_{i-1,j} - u_{i,j} + u_{i+1,j}$$

Using $j = 0$ in the above formula, we have

$$u_{i,1} = u_{i-1,0} - u_{i,0} + u_{i+1,0}$$

Computing the different values of $u_{i,1}$ for $i = 1, 2$ and 3, we get second row of following table

$j\ (t)$ \ $i\ (x)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.292803	0.414214	0.292803	0
2(1/8)	0				0

By using, $j = 1$ in the explicit formula, we get

$$u_{i,2} = u_{i-1,1} - u_{i,1} + u_{i+1,1}$$

For $i = 1, 2, 3$, we can find values at time $t = 1/8$. These values are given in the third row of the following table.

$i\,(x)$ / $j\,(t)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.292803	0.414214	0.292803	0
2(1/8)	0	0.121411	0.171392	0.121411	0

Exercice 04 :

Résolvez l'EDP de l'exercice 02 en utilisant le schéma Crank-Nicolson.

Solution :

Values of $u_{i,j}$ for $t = 1/16$ (or $j = 1$):

Crank–Nicolson formula (16.6) is as follows

$$-ru_{i+1,j+1} - ru_{i-1,j+1} + 2(1+r)u_{i,j+1} = ru_{i+1,j} + ru_{i-1,j} + 2(1-r)u_{i,j}$$

With $r = 1$, $j = 0$, the CN-formula gives following equations for $i = 1, 2, 3$

$$\begin{aligned} -u_{2,1} - u_{0,1} + 4u_{1,1} &= u_{2,0} + u_{0,0} \\ -u_{3,1} - u_{1,1} + 4u_{2,1} &= u_{3,0} + u_{1,0} \\ -u_{4,1} - u_{2,1} + 4u_{3,1} &= u_{4,0} + u_{2,0} \end{aligned} \tag{16.14}$$

Using values from Eqs. (16.12) and (16.13) in the system (16.14), we get

$$\begin{aligned} -u_{2,1} + 4u_{1,1} &= 1 + 0 = 1 \\ -u_{3,1} - u_{1,1} + 4u_{2,1} &= 1/\sqrt{2} + 1/\sqrt{2} = \sqrt{2} \\ -u_{2,1} + 4u_{3,1} &= 0 + 1 = 1 \end{aligned} \tag{16.15}$$

On solving the system (16.15), we have

$$u_{11} = u_{31} = 0.386729, \quad u_{21} = 0.546916 \tag{16.16}$$

$i\,(x)$ / $j\,(t)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.386729	0.546916	0.386729	0
2(1/8)	0				0

Values of $u_{i,j}$ for $t = 1/8$ (or $j = 2$):

With $r = 1$, $j = 1$, $i = 1, 2, 3$, the CN-formula (16.6) provides the following linear system

$$\begin{aligned} -u_{2,2} - u_{0,2} + 4u_{1,2} &= u_{2,1} + u_{0,1} \\ -u_{3,2} - u_{1,2} + 4u_{2,2} &= u_{3,1} + u_{1,1} \\ -u_{4,2} - u_{2,2} + 4u_{3,2} &= u_{4,1} + u_{2,1} \end{aligned} \tag{16.17}$$

Using the known values in the above system, and solving the resulting system of equations, we get

$$u_{12} = u_{32} = 0.211509, \quad u_{21} = 0.29912$$

$i\ (x)$ / $j\ (t)$	0(0)	1(1/4)	2(1/2)	3(3/4)	4(1)
0(0)	0	0.707107	1	0.707107	0
1(1/16)	0	0.386729	0.546916	0.386729	0
2(1/8)	0	0.211509	0.29912	0.211509	0

Référence bibliographique

[1] R.K. Gupta, *Numerical Methods: Fundamentals and Applications*, Cambridge University Press, 2019.